Transient analysis of an affine Queue-Hawkes process

Abstract

We investigate the transient behavior of a multi-server queue with a linearly increasing arrival rate, also called affine Queue-Hawkes process. We obtain expressions of the Laplace Transform of the transient probabilities involving confluent hypergeometric functions. When the arrival rate without congestion-attraction is higher than the service rate or with an initial condition with all servers busy, these Laplace Transforms can be numerically inverted. Our numerical investigations show that convergence to the stationary regime is slower than for an M/M/s queue.

Keywords: Queueing; transient analysis; Queue-Hawkes; Laplace Transform.

1 Introduction

[19] introduced the idea of self-excitement, a model in which the current intensity of events is determined by events in the past. In the so-called Hawkes processes, the rate of new event occurrences increases as each event occurs. The crowd-attraction behavior in Finance is well represented by Hawkes processes [13, 9, 18]. Hawkes processes are also known to capture the spread of infectious diseases [25], or the evolution of a population [7, 21, 28]. In contact centers also, [3] showed that the expressed sentiment of the customer influences agent response times and vice versa. The empirical analysis of [10] further proved that this phenomenon of self-excitement could be well modeled by Hawkes processes. In general, arrival processes with herding behavior in queueing systems can be modeled by Hawkes processes. For instance, [14] analyzed a queue with Hawkes arrival process and infinite number of servers in the asymptotic regime where the baseline intensity is large. The concept of Hawkes process has been extended to queueing processes via the definition of the Ephemerally Self-Exciting Process (ESEP) in [12]. In this paper, we analyze an ESEP with a linearly increasing arrival rate in the stationary and transient regimes. This model has been introduced in Section 3 of [12] and is called an *affine Queue-Hawkes Process*.

Specifically, we consider a single queue with a capacity of n customers and a pool of s homogeneous and independent servers, with $1 \leq s \leq n$. Service times are exponentially distributed with rate 1. Customers arrive at the system according to a Poisson process with state-dependent arrival rate $\lambda(x) = \lambda + \gamma x$, where x is the number of customers present in the system, for $0 \leq x < n$. If a customer is not routed to service immediately upon arrival, then she/he waits in a queue for her/his turn to be served, with customers being

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served in order of arrival. One way to interpret this arrival process is to consider a primary flow of *informed* customers who are insensitive to the system state and a secondary flow of arriving customers generated by the presence of customers in the system. Each customer present in the system generates a new Poisson arrival process with rate γ. Note that our model can be adjusted to the evolution of a population with the parameters s and n being infinite, γ being a birth rate, 1 being a mortality rate, and λ being an immigration rate. For the transient analysis, we let $N(t)$ be the number of customers in the system at time $t \geq 0$ and set $p_x(t) = P(N(t) = x | N(0) = x_0)$, so that $p_x(t)$ is the transient probability to have x customers present in the system at time t given that the system is in state x_0 at $t = 0$, for $0 \leq x_0 \leq n$. For the stationary regime, we denote by p_x the probability to have x customers in the system. When n is finite, the number of states is finite. So, the system is stable. When n is infinite, the stability condition is $s = \infty$ and $\gamma < 1$ [12] or $\lambda < s$ and $\gamma = 0$ as for an M/M/s queue.

Providing a transient analysis is useful for decision makers as the herding behavior seen in airports or in restaurants is often observed over short-time periods. Having a finite number of servers corresponds to situations where the system has a limited service capacity like in a restaurant or when a non-lethal disease can only be cured with hospital intervention. With an infinite number of servers, we mention [11] and [24] who determined the mean, and the moments of the Hawkes process-driven queue in both transient and steady state. Instead, in this paper, we express the Laplace Transform (LT) of the transient probabilities explicitly with finite or infinite number of servers and finite or infinite capacity. Having closed-form expressions is useful as it allows for a better understanding of the effect of the parameters and enables us to compute a specific probability without computing any other probabilities. Nevertheless, when the number of states is finite, other methods can be potentially more efficient to derive the transient probabilities. For instance, we mention the uniformization method introduced in [20] which was shown to be efficient in comparison with Runge-Kutta and Liou methods [15] and in comparison with Padé's approximation [30].

For our queueing model, in states where some servers are idling, Laplace Transforms of transient probabilities are expressed using hypergeometric functions when $\gamma \geq 1$. In the opposite case (as for the M/M/s queue), if the system starts when at least one server is idling, then the LT of transient probabilities are expressed using a contour integral. In states where all servers are busy, the LT of transient probabilities are expressed using the confluent hypergeometric functions of the first and second kind. Simplified expressions are provided when n or s tends to infinity. After applying a numerical inversion of the LT, we observe that the affine Queue-Hawkes process needs longer to reach stationarity compared to an M/M/s queue with the same flow of arrivals. This result is a consequence of over-dispersion of the arrival process. That is, the variance of the arrival process is larger than the mean, whereas the Poisson process has equal mean and variance. This phenomenon is also observed in other Hawkes processes [19] and further justifies the need to obtain transient expressions.

In what follows, we review the literature related to the analysis of transient queues. The M/M/1 queue was the first transient queue analyzed [26, 6]. The transient queue-length distribution for this queue is expressed through modified Bessel functions of the first kind. Later, [1] provided a transform factorization that helps developing approximations for the moments of the queue length in the M/M/1 queue. Several other approaches for the analysis of the M/M/1 queue have been developed. We refer to [31] for a review of the main findings for the computation of the performance measures of the M/M/1 queue. For the multi-server setting, [22] investigated the transient behavior of the M/M/s queue and showed the implications of their results for simulation. As in this paper, they considered the issue of the speed of convergence to the stationary regime. Later, [29] found a solution for the M/M/s queue from which the stationary regime could be derived. Including abandonment or rejection renders the performance evaluation difficult. We mention [2] and [23] for the performance measures of the M/M/s+M queue, [4] for the study of its busy period and [27] when deterministic reneging is involved.

Structure of the paper. The rest of the paper is organized as follows. Section 2 presents the stationary behavior of the affine Queue-Hawkes process. Section 3 determines the Laplace Transform of the transient probabilities. Section 4 provides a numerical illustration for the computation of the transient probabilities. The proofs are given in the appendix at the end of the paper.

2 Stationary analysis

We present the stationary behavior of our queueing model. We have $(\lambda + \gamma x)p_x = \min(x+1, s)p_{x+1}$, for $0 \leq x < n$. This leads to

$$p_x = p_0 \frac{\gamma^x \Gamma\left(\frac{\lambda}{\gamma} + x\right)}{x! \Gamma\left(\frac{\lambda}{\gamma}\right)}, \text{ for } 0 \leq x \leq s, \text{ and, } p_x = p_0 \frac{\gamma^x \Gamma\left(\frac{\lambda}{\gamma} + x\right)}{s! s^{x-s} \Gamma\left(\frac{\lambda}{\gamma}\right)}, \text{ for } s \leq x \leq n, \text{ with,} \tag{1}$$

$$p_0 = \left[\sum_{x=0}^{s-1} \frac{\gamma^x \Gamma\left(\frac{\lambda}{\gamma} + x\right)}{x! \Gamma\left(\frac{\lambda}{\gamma}\right)} + \sum_{x=s}^{n} \frac{\gamma^x \Gamma\left(\frac{\lambda}{\gamma} + x\right)}{s! s^{x-s} \Gamma\left(\frac{\lambda}{\gamma}\right)} \right]^{-1}, \tag{2}$$

where $\Gamma(z)$ is the Gamma-function defined for all complex numbers except the non-positive integers as $\Gamma(z) = \int_{t=0}^{\infty} t^{z-1} e^{-t} \, \mathrm{d}t$. Recall that our queueing model has been studied in [12] with an infinite number of servers or with a finite number of servers and no queue (i.e., the blocking model). As such, the stationary probabilities in [12], given in Theorem 3.3 in the infinite number of servers case can be deduced from our results in Equations (1) and (2) by letting s tend to infinity under the stability condition $\gamma < 1$. Those given

Table 1: Conditions for the monotonicity property of p_x in x ($s > 1$ and $n > s$)

Interval	p_x is increasing	p_x is decreasing	p_x has a minimum	p_x has a maximum
$0 \le x \le s$	$\lambda \ge 1$ and $\gamma \ge \frac{s-\lambda}{s-1}$	$\lambda \le 1$ and $\gamma \le \frac{s-\lambda}{s-1}$	$\lambda < 1$ and $\gamma > \frac{s-\lambda}{s-1}$	$\lambda > 1$ and $\gamma < \frac{s-\lambda}{s-1}$
$s \le x \le n$	$\gamma \ge \frac{s-\lambda}{s}$	$\gamma \le \frac{s-\lambda}{n-1}$	$\frac{s-\lambda}{n-1} < \gamma < \frac{s-\lambda}{s}$	-

in Section C for the blocking model can be obtained from Equations (1) and (2) by setting $s = n$.

Figure 1 presents the stationary probabilities in two systems with 10 servers and a capacity of 15 customers. In the first system, we set $\lambda = 1$ and $\gamma = 0.903$ to represent a situation with a significant congestion-attraction, and in the second one we adopt $\lambda = 6.803$ and $\gamma = 0$ to represent a queue without the phenomenon of congestion-attraction. The parameters are chosen such that the stationary expected number of customers in the system, $E(N)$, is equal to 7 in both cases. In the queue without congestion-attraction, the stationary

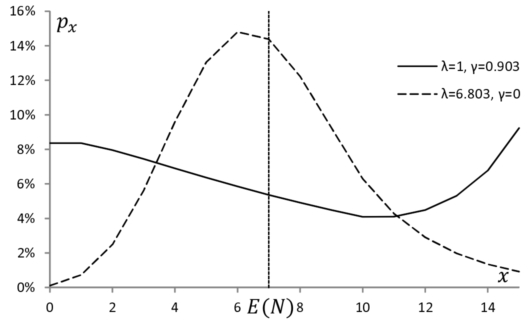


Figure 1: Stationary distribution ($s = 10$, $n = 15$)

probabilities are highest when they are close to $E(N)$. However, in the queue with congestion-attraction, extreme values ($x = 0$ and $x = n = 15$) have the highest probabilities. This indicates that this system spends a long time either being almost empty (i.e., around $x = 0$) or being highly congested (i.e., around $x = n$). In Table 1, we specify the conditions on the system parameters to have p_x increasing, decreasing, with one minimum, or with one maximum for the intervals $0 \le x \le s$ and $s \le x \le n$. This table is obtained by analyzing the sign of $\frac{p_{x+1}}{p_x} - 1$, for $s > 1$ and $n > s$. In the remaining case $s = 1$, we have $p_1 \ge p_0$ if and only if $\lambda \ge 1$. While [19] did not consider a queueing model, they also show that the Hawkes process is over-dispersed. This particular behavior of congestion-attraction queue, also viewed as an affine Queue-Hawkes process [12], slows down the speed of convergence to the stationary regime (see Section 3).

3 Transient analysis

We dedicate this section to the transient analysis of this queue. We provide an analysis involving complex integral to represent the Laplace Transforms (LT) of the transient probabilities as in [23]. This method is an alternative to the usual method which consists of finding the LT of the transient probabilities from the computation of the probability-generating function (see for instance [16] page 95) or the moment-generating function (see Section 3 of [12] for the affine Queue-Hawkes process similar to ours). This method enables us to directly express the solutions of the equations governing the evolution of the system state with known and tabulated confluent hypergeometric functions. The LT of the transient probabilities can be inverted either when the system starts with at least s customers or when the arrival rate generated by present customers (γ) is higher than or equal to the service rate.

The Markov chain associated with our queueing system can be split into two parts. In the first one, when the number of customers in the system is below s, the Markov chain behaves as in the evolution of a population with birth, death and immigration rates. The second part of the Markov chain, is the symmetric counterpart of the M/M/s+M queue. For this part of the Markov chain, the death rate is constant while the birth rate is linearly increasing. In the M/M/s+M queue, the birth rate is constant and the death rate is linearly increasing. Therefore, the method with contour integral adopted in [23] for the M/M/s+M queue provides us with interesting representations of the LT of the transient probabilities. As opposed to the M/M/s+M queue, the involved integrals are real and can be expressed with some known confluent hypergeometric functions. Analysis of the Markov chain is provided in Section 3.1. Section 3.2 combines the results of the previous section to explicitly state the LT of the transient probabilities.

We start the analysis with the equations governing the evolution of the system state. The forward Kolmogorov equations are

$$\begin{aligned}
&p_0'(t) = p_1(t) - \lambda p_0(t), \\
&p_x'(t) = (\lambda + (x-1)\gamma)p_{x-1}(t) - (\lambda + x\gamma + x)p_x(t) + (x+1)p_{x+1}(t), \text{ for } 1 \leq x \leq s-1, \\
&p_x'(t) = (\lambda + (x-1)\gamma)p_{x-1}(t) - (\lambda + x\gamma + s)p_x(t) + sp_{x+1}(t), \text{ for } s \leq x \leq n-1, \\
&p_n'(t) = (\lambda + (n-1)\gamma)p_{n-1}(t) - sp_n(t),
\end{aligned} \tag{3}$$

with $p_x(0) = \delta(x, x_0)$, where $\delta(x, x_0) = 1$ if $x = x_0$ and $\delta(x, x_0) = 0$, for $x \neq x_0$.

We set the Laplace transform of the transient probabilities as $P_x^*(\theta)= \int_0^\infty e^{-\theta t} p_x(t)\,\mathrm{d}t$, the system of Equations (3) can be rewritten as

$$P_1^*(\theta) - (\theta + \lambda)P_0^*(\theta) = -\delta(0, x_0), \tag{4}$$

$$(x+1)P_{x+1}^*(\theta) + (\lambda + (x-1)\gamma)P_{x-1}^*(\theta) - (\lambda + x(\gamma+1) + \theta)P_x^*(\theta) = -\delta(x, x_0), \text{for } 1 \leq x \leq s-1, \tag{5}$$

$$sP_{x+1}^*(\theta) + (\lambda + (x-1)\gamma)P_{x-1}^*(\theta) - (\lambda + x\gamma + s + \theta)P_x^*(\theta) = -\delta(x, x_0), \text{ for } s \leq x \leq n-1, \tag{6}$$

$$(\lambda + (n-1)\gamma)P_{n-1}^*(\theta) - (s + \theta)P_n^*(\theta) = -\delta(n, x_0). \tag{7}$$

We next solve Equations (4)-(7), by considering the cases $0 < x_0 < s$, $s < x_0 < n$, or $x_0 = 0, s, n$.

Remark: Equations (4)-(7) can also be employed to obtain the LT of the first passage time from one state to another. From the LT of the first passage time, we may easily derive the moments of the passage time from successive derivatives.

3.1 Solution of Equations (5) and (6)

In Theorem 1, we give the solutions of Equations (5) and (6). These solutions are constructed from contour integral where a generating function is introduced. We determine the differential equation that this generating function should satisfy and find two independent solutions. We denote by $F_x(\theta)$ and $G_x(\theta)$ the two independent solutions of Equation (5) and by $H_x(\theta)$ and $I_x(\theta)$ those of Equation (6). The solutions of these equations are given as functions of the confluent hypergeometric function of the first and second kind, $M(a,b,z)$ and $U(a,b,z)$ and the general hypergeometric function, $\Phi(a,b,c,z)$, for $a,b,c,z \in \mathbb{C}$, $Re(a) > 0$, with $|z| < 1$ and c not equal to a negative integer, where

$$\begin{aligned}
M(a,b,z) &= \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b)}{\Gamma(a)\Gamma(b+k)} \frac{z^k}{k!}, \; U(a,b,z) = \frac{1}{\Gamma(a)} \int_{u=0}^{\infty} e^{-zu} u^{a-1} (1+u)^{b-a-1}\, \mathrm{d}u, \text{ and,} \\
\Phi(a,b,c,z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)k!} z^k.
\end{aligned}$$

Theorem 1. *We have*

$$\begin{aligned}
F_x(\theta) &= \sum_{k=0}^{x} \frac{\gamma^k}{k!(x-k)!} \frac{\Gamma\left(\frac{\lambda}{\gamma} - \frac{\theta}{1-\gamma} + k\right)}{\Gamma\left(\frac{\lambda}{\gamma} - \frac{\theta}{1-\gamma}\right)} \frac{\Gamma\left(\frac{\theta}{1-\gamma} + x - k\right)}{\Gamma\left(\frac{\theta}{1-\gamma}\right)}, \; \textit{for } \gamma \neq 1, \\
F_x(\theta) &= \frac{\Gamma(x+\lambda)}{\Gamma(x+1)\Gamma(\lambda)} M(-x, \lambda, -\theta), \; \textit{for } \gamma = 1, \\
G_x(\theta) &= \gamma^{-\left(\frac{\lambda}{\gamma} + \frac{\theta}{\gamma-1}\right)} \frac{\Gamma\left(x + \frac{\lambda}{\gamma}\right)\Gamma\left(1 + \frac{\theta}{\gamma-1}\right)}{\Gamma\left(x + 1 + \frac{\lambda}{\gamma} + \frac{\theta}{\gamma-1}\right)} \Phi\left(\frac{\lambda}{\gamma} + \frac{\theta}{\gamma-1}, x + \frac{\lambda}{\gamma}, x + 1 + \frac{\lambda}{\gamma} + \frac{\theta}{\gamma-1}, \frac{1}{\gamma}\right), \; \textit{for } \gamma > 1, \\
G_x(\theta) &= \Gamma(x+\lambda) U(x+\lambda, \lambda, \theta), \; \textit{for } \gamma = 1, \\
G_x(\theta) &= \frac{1}{2\pi i} \int_{\tilde{\mathcal{C}}} z^{-(x+1)} (z-1)^{-\frac{\theta}{1-\gamma}} (\gamma z - 1)^{-\frac{\lambda}{\gamma} + \frac{\theta}{1-\gamma}}\, \mathrm{d}z, \; \textit{for } \gamma < 1, \\
H_x(\theta) &= \Gamma\left(\frac{\theta}{\gamma} + 1\right) e^{-\frac{s}{\gamma}} U\left(\frac{\theta}{\gamma} + 1, x + 1 + \frac{\lambda+\theta}{\gamma}, \frac{s}{\gamma}\right), \; \textit{and} \\
I_x(\theta) &= M\left(x + \frac{\lambda}{\gamma}, x + 1 + \frac{\lambda+\theta}{\gamma}, -\frac{s}{\gamma}\right) \frac{\Gamma\left(x + \frac{\lambda}{\gamma}\right)\Gamma\left(1 + \frac{\theta}{\gamma}\right)}{\Gamma\left(x + 1 + \frac{\lambda+\theta}{\gamma}\right)},
\end{aligned}$$

where $\tilde{\mathcal{C}}$ is a contour which starts at $-\infty - i\epsilon$ and ends at $-\infty + i\epsilon$ encircling $z = 0$, $z = \frac{1}{\gamma}$, and $z = 1$ in the counterclockwise sense.

With this definition, we obtain $F_{-1}(\theta) = 0$, $F_0(\theta) = 1$, and $F_1(\theta) = \theta + \lambda$. Therefore, Equation (4) is also satisfied by $F_x(\theta)$ but not by $G_x(\theta)$. In the case of $\gamma < 1$, it is not possible to express $G_x(\theta)$ explicitly or as a function of some known functions.

Model extension: $\lambda(x) = \lambda + \gamma \min(x, c)$**, for** $0 \leq c \leq n$**.** The analysis with state-dependent arrival rate $\lambda(x) = \lambda + \gamma x$ considered in this paper can be extended to the case $\lambda(x) = \lambda + \gamma \min(x, c)$, for $0 \leq c \leq n$ to account for bounded arrival rates. The following intervals should be considered:

- For $0 \leq x \leq \min(c, s)$, Equation (5) is unchanged. So, $P_x(\theta)$ can be expressed as a linear combination of $F_x(\theta)$ and $G_x(\theta)$ as expressed in Theorem 1.

- If $c \leq s$ and $c \leq x \leq s$, then Equation (5) is modified by letting first γ tend to zero and next replace λ by $\lambda + c\gamma$. Observe that $(1-\gamma t)^{-\frac{\lambda}{\gamma}+\frac{\theta}{1-\gamma}}$ tends to $e^{\lambda t}$ as γ tends to zero. Therefore, from the analysis in the proof of Theorem 1, $P_x(\theta)$ can be written as a linear combination of $\overline{F_x}(\theta)$ and $\overline{G_x}(\theta)$ defined as $\overline{F_x}(\theta) = \sum_{k=0}^{x} \frac{(\lambda+c\gamma)^k}{k!(x-k)!} \frac{\Gamma(\theta+x-k)}{\Gamma(\theta)}$ and $\overline{G_x}(\theta) = \dfrac{1}{2\pi i} \displaystyle\int_{\tilde{\mathcal{C}}} z^{-(x+1)} (z-1)^{-\theta} e^{(\lambda+c\gamma)z} \, \mathrm{d}z$.

- If $c \geq s$ and $s \leq x \leq c$, Equation (6) is unchanged. Thus, $P_x(\theta)$ can be expressed as a linear combination of $H_x(\theta)$ and $I_x(\theta)$ as expressed in Theorem 1.

- For $\max(c, s) \leq x \leq n$, the transitions are identical to those of an M/M/1 queue. Therefore, the results in [17] page 99 apply and $P_x(\theta)$ can be expressed as a linear combination of $\left(\frac{\lambda+c\gamma+s+\theta+\sqrt{(\lambda+c\gamma+s+\theta)^2-4(\lambda+c\gamma)s}}{2(\lambda+c\gamma)}\right)^x$ and $\left(\frac{\lambda+c\gamma+s+\theta-\sqrt{(\lambda+c\gamma+s+\theta)^2-4(\lambda+c\gamma)s}}{2(\lambda+c\gamma)}\right)^x$.

3.2 Expressions of the transient probabilities

We are now in position to express the Laplace Transform of the transient probabilities. These expressions, given in Theorem 2 with $\lambda(x) = \lambda + \gamma x$ for $0 \leq x \leq n$, involve the Wronskian of the sequences $F_x(\theta), G_x(\theta), H_x(\theta)$ and $I_x(\theta)$. We define the Wronskian application W for two sequences $U_x(\theta)$ and $V_x(\theta)$, such that $W_x^{(U,V)}(\theta) = U_{x+1}(\theta)V_x(\theta) - V_{x+1}(\theta)U_x(\theta)$.

Theorem 2. *We have with $0 \le x_0 < s$,*

$$
\begin{aligned}
P_x^*(\theta) &= \frac{\frac{W_{s-1}^{(G,H)}(\theta)-\alpha_n(\theta)W_{s-1}^{(G,I)}}{W_{s-1}^{(F,H)}(\theta)-\alpha_n(\theta)W_{s-1}^{(F,I)}(\theta)}F_{x_0}(\theta)-G_{x_0}(\theta)}{(x_0+1)W_{x_0}^{(G,F)}(\theta)}F_x(\theta), \ \textit{for}, \ 0 \le x \le x_0, \\
P_x^*(\theta) &= \frac{F_{x_0}(\theta)}{(x_0+1)W_{x_0}^{(G,F)}(\theta)}\left(\frac{W_{s-1}^{(G,H)}(\theta)-\alpha_n(\theta)W_{s-1}^{(G,I)}(\theta)}{W_{s-1}^{(F,H)}(\theta)-\alpha_n(\theta)W_{s-1}^{(F,I)}(\theta)}F_x(\theta)-G_x(\theta)\right), \ \textit{for}, \ x_0 \le x \le s, \ \textit{and}, \\
P_x^*(\theta) &= \frac{F_{x_0}(\theta)}{W_{s-1}^{(F,H)}(\theta)-\alpha_n(\theta)W_{s-1}^{(F,I)}(\theta)}\frac{\gamma^{s-x_0-1}x_0!}{s!}\frac{\Gamma\left(\frac{\lambda}{\gamma}+s-1\right)}{\Gamma\left(\frac{\lambda}{\gamma}+x_0\right)}\left(H_x(\theta)-\alpha_n(\theta)I_x(\theta)\right), \ \textit{for}, \ s \le x \le n.
\end{aligned}
$$

For $s \le x_0 \le n$, we have

$$
\begin{aligned}
P_x^*(\theta) &= \frac{\left(\frac{\gamma}{s}\right)^{s-x_0-1}\Gamma\left(\frac{\lambda}{\gamma}+s-1\right)}{s\Gamma\left(\frac{\lambda}{\gamma}+x_0\right)}\frac{\alpha_n(\theta)I_{x_0}(\theta)-H_{x_0}(\theta)}{W_{s-1}^{(H,F)}(\theta)-\alpha_n(\theta)W_{s-1}^{(I,F)}(\theta)}F_x(\theta), \ \textit{for}, \ 0 \le x \le s, \\
P_x^*(\theta) &= \frac{\alpha_n(\theta)I_{x_0}(\theta)-H_{x_0}(\theta)}{sW_{x_0}^{(I,H)}(\theta)(W_{s-1}^{(H,F)}(\theta)-\alpha_n(\theta)W_{s-1}^{(I,F)}(\theta))}(W_{s-1}^{(I,F)}(\theta)H_x(\theta)-W_{s-1}^{(H,F)}(\theta)I_x(\theta)), \ \textit{for}, \ s \le x \le x_0, \ \textit{and}, \\
P_x^*(\theta) &= \frac{I_{x_0}(\theta)W_{s-1}^{(H,F)}(\theta)-H_{x_0}(\theta)W_{s-1}^{(I,F)}(\theta)}{sW_{x_0}^{(I,H)}(\theta)(W_{s-1}^{(H,F)}(\theta)-\alpha_n(\theta)W_{s-1}^{(I,F)}(\theta))}\left(H_x(\theta)-\alpha_n(\theta)I_x(\theta)\right), \ \textit{for}, \ x_0 \le x \le n,
\end{aligned}
$$

where $\alpha_n(\theta) = \frac{sH_{n+1}(\theta)-(\lambda+n\gamma)H_n(\theta)}{sI_{n+1}(\theta)-(\lambda+n\gamma)I_n(\theta)}$.

In asymptotic cases, the formulas of Theorem 2 can be simplified. In what follows, we present the cases of $n = \infty$ and $s = n = \infty$.

Infinite capacity queue. We first consider the case with an infinite capacity queue (i.e., $n = \infty$). The analysis of Theorem 2 can be made in a similar way using $\lim_{x\to\infty} I_x = 0$ and $\lim_{x\to\infty} H_x = \infty$. With $0 \le x_0 \le s$, we have

$$
\begin{aligned}
P_x^*(\theta) &= \frac{F_{x_0}(\theta)W_{s-1}^{(G,I)}(\theta)-G_{x_0}(\theta)W_{s-1}^{(F,I)}(\theta)}{(x_0+1)W_{x_0}^{(G,F)}(\theta)W_{s-1}^{(F,I)}(\theta)}F_x(\theta), \ \text{for } 0 \le x \le x_0, \\
P_x^*(\theta) &= \frac{F_{x_0}(\theta)}{(x_0+1)W_{x_0}^{(G,F)}(\theta)}\left(\frac{W_{s-1}^{(G,I)}(\theta)}{W_{s-1}^{(F,I)}(\theta)}F_x(\theta)-G_x(\theta)\right), \ \text{for } x_0 \le x \le s, \ \text{and}, \\
P_x^*(\theta) &= \frac{F_{x_0}(\theta)}{W_{s-1}^{(F,I)}(\theta)}\frac{\gamma^{s-x_0-1}x_0!\Gamma\left(\frac{\lambda}{\gamma}+s-1\right)}{s!\Gamma\left(\frac{\lambda}{\gamma}+x_0\right)}I_x(\theta), \ \text{for } x \ge s.
\end{aligned}
$$

With $x_0 \geq s$, we get

$$P_x^*(\theta) = \frac{I_{x_0}(\theta)}{sW_{s-1}^{(F,I)}(\theta)} \frac{\left(\frac{\gamma}{s}\right)^{s-1-x_0} \Gamma\left(\frac{\lambda}{\gamma}+s-1\right)}{\Gamma\left(\frac{\lambda}{\gamma}+x_0\right)} F_x(\theta), \text{ for } 0 \leq x \leq s,$$

$$P_x^*(\theta) = \frac{I_{x_0}(\theta)}{sW_{x_0}^{(H,I)}(\theta)} \left(H_x(\theta) - \frac{W_{s-1}^{(F,H)}(\theta)}{W_{s-1}^{(F,I)}(\theta)} I_x(\theta)\right), \text{ for } s \leq x \leq x_0, \text{ and,}$$

$$P_x^*(\theta) = \frac{W_{s-1}^{(F,I)}(\theta) H_{x_0}(\theta) - W_{s-1}^{(F,H)}(\theta) I_{x_0}(\theta)}{sW_{s-1}^{(F,I)}(\theta) W_{x_0}^{(H,I)}(\theta)} I_x(\theta), \text{ for } x \geq x_0.$$

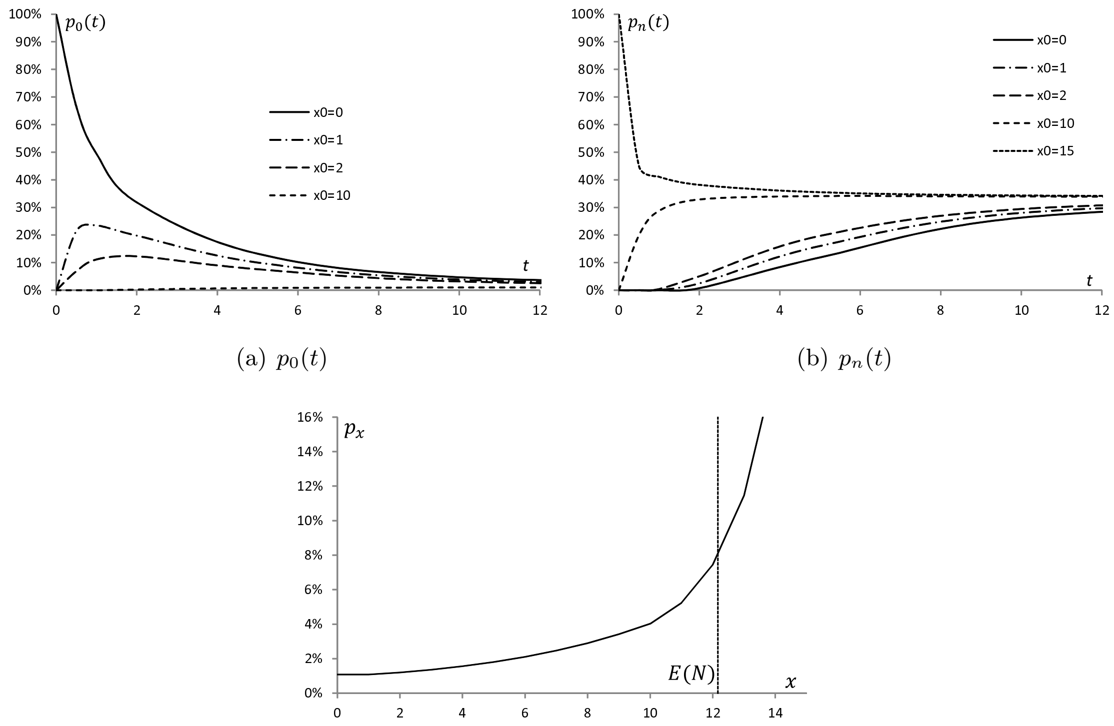
Infinite number of servers. In the case with an infinite number of servers and an infinite capacity, the formulas can be further simplified into

$$P_x^*(\theta) = \frac{G_{x_0}(\theta)}{(x_0+1)W_{x_0}^{(F,G)}(\theta)} F_x(\theta), \text{ for } 0 \leq x \leq x_0, \text{ and, } P_x^*(\theta) = \frac{F_{x_0}(\theta)}{(x_0+1)W_{x_0}^{(F,G)}(\theta)} G_x(\theta), \text{ for } x \geq x_0.$$

In the case $x_0 = 0$ and $\gamma = 1$, we get $P_x^*(\theta) = \frac{G_x(\theta)}{(1+\theta)G_0(\theta) - G_1(\theta)}$, where $G_x(\theta) = \int_{t=0}^{\infty} e^{-\theta t} \frac{t^{x-1+\lambda}}{(1+t)^{x+1}} \mathrm{d}t$, for $x \geq 0$. Using an integration by part, we obtain $G_1(\theta) = -\theta \int_{t=0}^{\infty} e^{-\theta t} \frac{t^{\lambda}}{1+t} \mathrm{d}t + \lambda G_0(\theta)$. So, $(\lambda+\theta)G_0(\theta) - G_1(\theta) = \theta \int_{t=0}^{\infty} e^{-\theta t} t^{\lambda-1} \mathrm{d}t = \theta^{1-\lambda}\Gamma(\lambda)$. Finally, we obtain $P_x^*(\theta) = \frac{\theta^{\lambda-1}}{\Gamma(\lambda)} \int_{t=0}^{\infty} e^{-\theta t} \frac{t^{x-1+\lambda}}{(1+t)^{x+1}} \mathrm{d}t$. In the case $\lambda = 1$, we deduce that $p_x(t) = \frac{t^x}{(1+t)^{x+1}}$. This means that $p_x(t)$ has a geometric distribution. Since $\lim_{t\to\infty} p_x(t) = 0$, the combination of parameters $\lambda = \gamma = 1$ and $s = \infty$ leads to an unstable system.

4 Numerical illustration

The transient performance measures can be computed using a Laplace transform inversion. We use the speed up version of the Gaver-Stehfest algorithm presented in [8], page 144, equation (7.7), where a given function $f(t)$ is approximated by $\frac{\ln(2)}{t} \sum_{j=1}^{N} K_j \cdot f^*\left(j\frac{\ln(2)}{t}\right)$, where N is even and $K_j = (-1)^{j+\frac{N}{2}} \sum_{k=\left[\frac{j+1}{2}\right]}^{\min(j,N/2)} \frac{k^{N/2}(2k)!}{(N/2-k)!k!(k-1)!(j-k)!(2k-j)!}$. In Figures 2 and 3, we derive $p_0(t)$ and $p_n(t)$ as functions of the time elapsed since the origin, t, for different values of the initial state. Figure 2 presents a situation where the stationary probabilities are strictly increasing in the system state while Figure 3 illustrates a case where extreme states have the highest stationary probabilities. When comparing Figures 2(a) and 2(b) with Figures 3(a) and 3(b), we observe that the speed of convergence to the stationary distribution is slower in Figure 3. In Figure 2, the state $x = n = 15$ plays a role of attraction in the evolution of the system state. In Figure 3, both states $x = 0$ and $x = n = 15$ play this role. This means in practice that the system state remains either around $x = 0$ or around $x = 15$ for a long time, which slows down the achievement of a close to stationary regime. This means that a system



(a) $p_0(t)$

(b) $p_n(t)$

(c) Stationary distribution

Figure 2: Numerical results ($\lambda = 1$, $\gamma = 1.2$, $s = 10$, $n = 15$, $N = 50$)

with congestion-attraction may take a long time before reaching the stationary regime as compared to classical queueing models used to represent service systems. It also means that system managers should be careful while using stationary performance measures when making staffing or routing decisions over a short-term horizon. For instance, if the system initially starts empty, stationary performance measures could lead to over-staffing decisions if the congestion-attraction phenomenon is not observed for a sufficiently long time.

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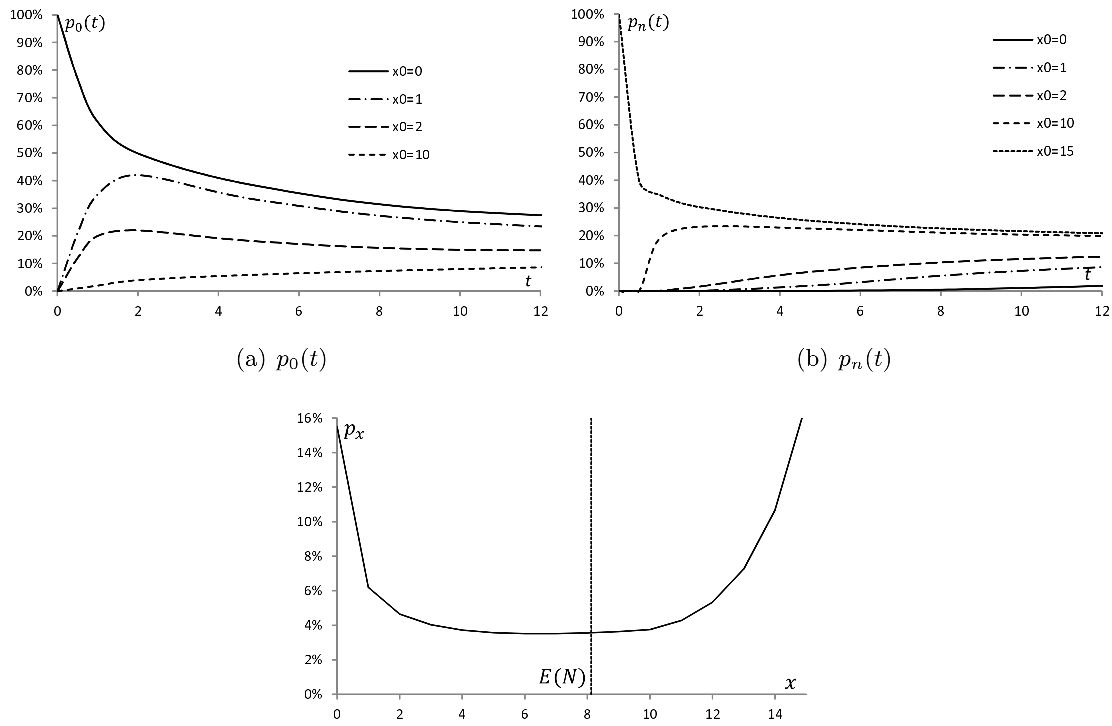

(a) $p_0(t)$

(b) $p_n(t)$

(c) Stationary distribution

Figure 3: Numerical results ($\lambda = 0.4$, $\gamma = 1.1$, $s = 10$, $n = 15$, $N = 50$)

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A Proof of Theorem 1

Proof. We first consider Equation (5). We propose to build two independent solutions to this equation. We introduce the sequence f_x which satisfies $(x+1)f_{x+1} + (\lambda + (x-1)\gamma)f_{x-1} = (\lambda + x(\gamma+1) + \theta)f_x$. The idea is to propose to write f_x as an integral and to determine the differential equation that the generating function $\mathcal{F}(z)$ should satisfy in order to generate some solutions to Equation (5). From Cauchy' integral theorem, the function f_x can be written as $f_x = \int_{\mathcal{C}} z^{-(x+1)}\mathcal{F}(z)\,\mathrm{d}z$, for some contour $\mathcal{C}$ and some function $\mathcal{F}(z)$. Then, we have

$$\int_{\mathcal{C}} z^{-(x+1)} \left(\lambda + x(\gamma+1) + \theta - (\lambda + (x-1)\gamma)z - \frac{x+1}{z}\right) \mathcal{F}(z)\,\mathrm{d}z = 0. \tag{8}$$

We assume that $\mathcal{C}$ is such that there are no boundary contributions arising from endpoints of $\mathcal{C}$ in order to apply the integration by parts and write $xf_x = \int_{\mathcal{C}} xz^{-(x+1)}\mathcal{F}(z)\,\mathrm{d}z = \int_{\mathcal{C}} z^{-x}\mathcal{F}'(z)\,\mathrm{d}z$. Therefore, Equation (8) leads to

$$\int_{\mathcal{C}} z^{-(x+1)} \left[-(1-z)(1-\gamma z)\mathcal{F}'(z) + (\lambda(1-z) + \theta)\,\mathcal{F}(z)\right]\,\mathrm{d}z = 0. \tag{9}$$

One solution of this equation can be obtained if for all z we have

$$(1-z)(1-\gamma z)\mathcal{F}'(z) = (\lambda(1-z)+\theta)\,\mathcal{F}(z). \tag{10}$$

One solution of Equation (10) is

$$\mathcal{F}(z) = (1-z)^{-\frac{\theta}{1-\gamma}}(1-\gamma z)^{-\frac{\lambda}{\gamma}+\frac{\theta}{1-\gamma}},\ \text{if } \gamma \neq 1,\ \text{and, } \mathcal{F}(z) = (1-z)^{-\lambda}e^{\frac{\theta}{1-z}},\ \text{if } \gamma = 1.$$

Note that $\mathcal{F}(x)$ has branch points at $z=1$ if $\gamma<1$ and at $z=1/\gamma$ if $-\frac{\lambda}{\gamma}+\frac{\theta}{1-\gamma}<0$. Based on $\mathcal{F}(z)$, we are in position to build two independent solutions for Equation (5) denoted by $F_x(\theta)$ and $G_x(\theta)$.

The function $F_x(\theta)$.

Case 1: $\gamma \neq 1$. We consider the integral $F_x(\theta)= \dfrac{1}{2\pi i}\displaystyle\int_{\mathcal{C}_1} z^{-(x+1)}(1-z)^{-\frac{\theta}{1-\gamma}}(1-\gamma z)^{-\frac{\lambda}{\gamma}+\frac{\theta}{1-\gamma}}\,\mathrm{d}z$, where $\mathcal{C}_1$ is a small circle in the $z-$plane on which $|z|<\min\left(1,\frac{1}{\gamma}\right)$. The integrand in the expression of $F_x(\theta)$ is analytic inside the circle $\mathcal{C}_1$. We can obtain an explicit expression of F_x. Since $(1-z)^{-\frac{\theta}{1-\gamma}} = \sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\theta}{1-\gamma}+k\right)}{\Gamma\left(\frac{\theta}{1-\gamma}\right)k!}z^k$, and $(1-\gamma z)^{-\frac{\lambda}{\gamma}+\frac{\theta}{1-\gamma}} = \sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{\lambda}{\gamma}-\frac{\theta}{1-\gamma}+k\right)}{\Gamma\left(\frac{\lambda}{\gamma}-\frac{\theta}{1-\gamma}\right)k!}\gamma^k z^k$, we have

$$(1-z)^{-\frac{\theta}{1-\gamma}}(1-\gamma z)^{-\frac{\lambda}{\gamma}+\frac{\theta}{1-\gamma}} = \sum_{m=0}^{\infty} z^m\left(\sum_{k=0}^{m}\frac{\gamma^k}{k!(m-k)!}\frac{\Gamma\left(\frac{\lambda}{\gamma}-\frac{\theta}{1-\gamma}+k\right)}{\Gamma\left(\frac{\lambda}{\gamma}-\frac{\theta}{1-\gamma}\right)}\frac{\Gamma\left(\frac{\theta}{1-\gamma}+m-k\right)}{\Gamma\left(\frac{\theta}{1-\gamma}\right)}\right).$$

This leads to the expression of $F_x(\theta)$ as in Theorem 1.

Case 2: $\gamma = 1$. We define $F_x(\theta)$ as $F_x(\theta)= \dfrac{e^{-\theta}}{2\pi i}\displaystyle\int_{\mathcal{C}_1} z^{-(x+1)}(1-z)^{-\lambda}e^{\frac{\theta}{1-z}}\,\mathrm{d}z$, where $\mathcal{C}_1$ is defined as above. The definition of $F_x(\theta)$ corresponds to the integral representation of the Laguerre polynomials. Using Equation (10.50) page 399 in [5], we may then write $F_x(\theta) = \frac{\Gamma(x+\lambda)}{\Gamma(x+1)\Gamma(\lambda)}M(-x,\lambda,-\theta)$, where $M(a,b,z)$ is the confluent hypergeometric function of the first kind.

The function $G_x(\theta)$. Due to the initial condition at $t=0$ we need another solution to Equation (5) (independent from $F_x(\theta)$). We differentiate the cases $\gamma>1$, $\gamma=1$, and $\gamma<1$.

Case 1: $\gamma > 1$. In this case, we observe that $\mathcal{F}(1)=0$, we can then define G_x as an integral on the real interval $(1,\infty)$ by $G_x(\theta)= \displaystyle\int_{z=1}^{\infty} z^{-(x+1)}(z-1)^{\frac{\theta}{\gamma-1}}(\gamma z-1)^{-\left(\frac{\lambda}{\gamma}+\frac{\theta}{\gamma-1}\right)}\,\mathrm{d}z$. We change the variable z by $t=1/z$. This leads to $G_x(\theta)= \displaystyle\int_{t=0}^{1}\gamma^{-\left(\frac{\lambda}{\gamma}+\frac{\theta}{\gamma-1}\right)}t^{x-1+\frac{\lambda}{\gamma}}(1-t)^{\frac{\theta}{\gamma-1}}\left(1-\frac{t}{\gamma}\right)^{-\left(\frac{\lambda}{\gamma}+\frac{\theta}{\gamma-1}\right)}\,\mathrm{d}t$. Therefore, G_x can be related to the general hypergeometric function. This function has an integral representation when $Re(c) > Re(b) > 0$; $\Phi(a,b,c,z)= \dfrac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \displaystyle\int_{t=0}^{1} t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t$, which proves the expression of $G_x(\theta)$.

Case 2: $\gamma = 1$. In this case, we observe that $\mathcal{F}(1^+) = 0$, we can then define G_x as an integral on the real interval $(1,\infty)$ by $G_x(\theta)= \displaystyle\int_{z=1}^{\infty} z^{-(x+1)}(z-1)^{-\lambda}e^{\frac{\theta}{1-z}}\,\mathrm{d}z$. We change the variable z in $u = \frac{-1}{1-z}$. This leads to $G_x(\theta)= \displaystyle\int_{u=0}^{\infty} u^{x-1+\lambda}(1+u)^{-(x+1)}e^{-\theta u}\,\mathrm{d}u$. Thus, we may write $G_x(\theta) = \Gamma(x+\lambda)U(x+\lambda,\lambda,\theta)$.

Case 3: $\gamma < 1$. This case is more difficult since we may not find z such that $\mathcal{F}(z) = 0$. We instead consider the complex integral with contour $\tilde{\mathcal{C}}$ which starts at $-\infty - i\epsilon$ and ends at $-\infty + i\epsilon$ encircling $z = 0$, $z = \frac{1}{\gamma}$, and $z = 1$ in the counterclockwise sense. We thus define G_x as $G_x(\theta)= \dfrac{1}{2\pi i}\displaystyle\int_{\mathcal{C}_2} z^{-(x+1)}(z-1)^{-\frac{\theta}{1-\gamma}}(\gamma z-1)^{-\frac{\lambda}{\gamma}+\frac{\theta}{1-\gamma}}\,\mathrm{d}z$. $G_x(\theta)$ satisfies the homogeneous form of Equation (5).

We now consider Equation (6) with the same approach. We shall construct two independent solutions to this difference equation. Let g_x satisfy $sg_{x+1} + (\lambda + (x-1)\gamma)g_{x-1} = (\lambda + x\gamma + s + \theta)g_x$ and represent g_x as a contour integral, with $g_x= \displaystyle\int_{\mathcal{C}} z^{-(x+1)}\mathcal{G}(z)\,\mathrm{d}z$, for some function $\mathcal{G}(.)$ and contour $\mathcal{C}$. Then, we have

$$\int_{\mathcal{C}} z^{-(x+1)}\left(\lambda + x\gamma + s + \theta - (\lambda + (x-1)\gamma)z - \frac{s}{z}\right)\mathcal{G}(z)\,\mathrm{d}z = 0. \tag{11}$$

Again, we assume that $\mathcal{C}$ is such that there are no boundary contributions arising from endpoints of $\mathcal{C}$ in order to apply the integration by parts and obtain $\displaystyle\int_{\mathcal{C}} z^{-(x+1)}\left[\gamma z(1-z)\mathcal{G}'(z) + \left(\lambda(1-z) + s + \theta - \frac{s}{z}\right)\mathcal{G}(z)\right]\mathrm{d}z = 0$. One solution of the above equation can be obtained if for all z we have

$$\gamma z(1-z)\mathcal{G}'(z) = -\left(\lambda(1-z) + s + \theta - \frac{s}{z}\right)\mathcal{G}(z). \tag{12}$$

One solution of Equation (12) is

$$\mathcal{G}(z) = e^{-\frac{s}{\gamma z}}z^{-\frac{\lambda+\theta}{\gamma}}(1-z)^{\frac{\theta}{\gamma}}. \tag{13}$$

The function $\mathcal{G}(z)$ has a unique branch point at $z = 0$. To determine two independent solutions of Equation (6), we shall find two intervals leading to two independent solutions. These functions are denoted by $H_x(\theta)$ and $I_x(\theta)$.

The function $H_x(\theta)$**.** Since $\mathcal{G}(1) = \mathcal{G}(0^+) = 0$, we first consider the interval $(0,1)$. Thus, we define $H_x(\theta)$ as $H_x(\theta) = \displaystyle\int_{t=0}^{1} e^{-\frac{s}{\gamma t}}t^{-(x+1)-\frac{\lambda+\theta}{\gamma}}(1-t)^{\frac{\theta}{\gamma}}\,\mathrm{d}t$. With the change of variable $u = \frac{1}{t} - 1$, we obtain $H_x(\theta) = e^{-\frac{s}{\gamma}}\displaystyle\int_{u=0}^{\infty} e^{-\frac{s}{\gamma}u}(u+1)^{x-1+\frac{\lambda}{\gamma}}u^{\frac{\theta}{\gamma}}\,\mathrm{d}u$, which leads to the expression of $H_x(\theta)$.

The function $I_x(\theta)$. We now define $I_x(\theta)$ as $I_x(\theta) = \displaystyle\int_{t=1}^{\infty} e^{-\frac{s}{\gamma t}} t^{-(x+1)-\frac{\lambda+\theta}{\gamma}} (t-1)^{\frac{\theta}{\gamma}} \,\mathrm{d}t$. With the change of variable $z = \frac{1}{t}$, we obtain $I_x(\theta) = \displaystyle\int_{z=0}^{1} e^{-\frac{s}{\gamma} z} z^{x-1+\frac{\lambda}{\gamma}} (1-z)^{\frac{\theta}{\gamma}} \,\mathrm{d}z$. Since $M(a,b,z)$ has the following integral representation $M(a,b,z) = \dfrac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \displaystyle\int_{u=0}^{1} e^{zu} u^{a-1} (1-u)^{b-a-1} \,\mathrm{d}u$, for $Re(b) > Re(a) > 0$, we obtain the expression of $I_x(\theta)$. $\square$

B Proof of Theorem 2

Proof. To simplify the writing, we do not express the dependence of the considered functions in θ in the proof. We first consider the case $0 < x_0 < s$. We introduce the constant A, B, C, D, and E such that

$$P_x^* = AF_x, \text{ for, } 0 \le x \le x_0, \; P_x^* = BF_x + CG_x, \text{ for, } x_0 \le x \le s, \text{ and } P_x^* = DH_x + EI_x, \text{ for, } s \le x \le n.$$

The boundary equations allow us to relate the different constants. With Equation (7), we have $(\lambda + (n-1)\gamma)(DH_{n-1} + EI_{n-1}) - (s+\theta)(DH_n + EI_n) = 0$. Moreover, we have $s(DH_{n+1} + EI_{n+1}) + (\lambda + (n-1)\gamma)(DH_{n-1} + EI_{n-1}) - (\lambda + n\gamma + s + \theta)(DH_n + EI_n) = 0$. By subtracting these two equations, we get $s(DH_{n+1} + EI_{n+1}) - (\lambda + n\gamma)(DH_n + EI_n) = 0$. Therefore,

$$E = -D \frac{sH_{n+1} - (\lambda + n\gamma)H_n}{sI_{n+1} - (\lambda + n\gamma)I_n} = -D\alpha_n.$$

For $x = s$, we may write

$$BF_s + CG_s = D(H_s - \alpha_n I_s), \text{ and, } (s+1)(BF_{s+1} + CG_{s+1}) = sD(H_{s+1} - \alpha_n I_{s+1}).$$

This leads to $B = D\frac{(s+1)G_{s+1}(H_s - \alpha_n I_s) - sG_s(H_{s+1} - \alpha_n I_{s+1})}{(s+1)(F_s G_{s+1} - G_s F_{s+1})} = D\beta_{s,n}^1$, and
$C = -D\frac{(s+1)F_{s+1}(H_s - \alpha_n I_s) - sF_s(H_{s+1} - \alpha_n I_{s+1})}{(s+1)(F_s G_{s+1} - G_s F_{s+1})} = -D\beta_{s,n}^2$. We now use the condition $x = x_0$. This leads to

$$AF_{x_0} = D\left(\beta_{s,n}^1 F_{x_0} - \beta_{s,n}^2 G_{x_0}\right), \text{ and, } A(x_0+1)F_{x_0+1} = D(x_0+1)\left(\beta_{s,n}^1 F_{x_0+1} - \beta_{s,n}^2 G_{x_0+1}\right) + 1.$$

Therefore, we have $A = \frac{\beta_{s,n}^1 F_{x_0} - \beta_{s,n}^2 G_{x_0}}{\beta_{s,n}^2 (x_0+1)(F_{x_0} G_{x_0+1} - G_{x_0} F_{x_0+1})}$, and, $D = \frac{F_{x_0}}{\beta_{s,n}^2 (x_0+1)(F_{x_0} G_{x_0+1} - G_{x_0} F_{x_0+1})}$. Using Equation (5), we find that $(x+1)W_x^{(G,F)} = (\lambda + (x-1)\gamma)W_{x-1}^{(G,F)}$. Therefore, we deduce that $W_x^{(G,F)} = W_0^{(G,F)} \frac{\gamma^x \Gamma\left(\frac{\lambda}{\gamma} + x\right)}{(x+1)!\Gamma\left(\frac{\lambda}{\gamma}\right)}$, which further simplifies the expressions. The formulas for $0 < x_0 < s$ can be extended to the case $x_0 = 0$. The case $s < x_0 < n$ can be obtained with a similar approach. We introduce the constant $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, and $\tilde{E}$ such that $P_x^* = \tilde{A}F_x$, for, $0 \le x \le s$, $P_x^* = \tilde{B}H_x + \tilde{C}I_x$, for, $s \le x \le x_0$, and $P_x^* = \tilde{D}H_x + \tilde{E}I_x$, for, $x_0 \le x \le n$. One difference here is that the function G_x is not involved in the solution. After using the boundary equations, the Wronskian of I_x and H_x is involved. We find that $sW_x^{(I,H)} = (\lambda + (x-1)\gamma)W_{x-1}^{(I,H)}$. Therefore, we deduce that $W_x^{(I,H)} = W_0^{(I,H)} \frac{\left(\frac{\gamma}{s}\right)^x \Gamma\left(\frac{\lambda}{\gamma}+x\right)}{\Gamma\left(\frac{\lambda}{\gamma}\right)}$. The formulas obtained for $s < x_0 < n$ are valid for $x_0 = s$ and for $x_0 = n$. $\square$