A Time-based Policy for Empty Container Management by Consignees

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Abstract

Nowadays, the majority of consumer goods is transported into a maritime container during at least one stage of the journey. Besides the many advantages of containerization, the management of empty containers is a key issue responsible for costly repositioning operations. This article investigates the potential for consignees to manage an inventory of empty containers at their location so as to enable direct reuse of these containers by shippers located in the surroundings. The complexity in developing a policy for empty container management by consignees results from the non-linearity of detention costs imposed by shipping companies under merchant haulage combined with fixed repositioning costs. We formulate the problem as a Markov decision process using the waiting time of the oldest container as a decision variable. Next, we use value iteration to prove that a threshold policy in the age of the oldest container in stock is optimal among the class of time-based policies. We derive closed-form formulas for the system performance under the optimal policy. This allows us to compute numerically the optimal threshold and to derive explicit expressions of the optimal threshold in asymptotic regimes. We next analyze the impact of this proactive management of empty containers by the consignees on the level of direct container reuse. We show that this practice is very promising to enable a high level of direct reuse. This practice also enables the consignee to reduce container repositioning costs but the incentive to implement the policy we propose varies a lot from one setting to another. So, we further explore if the incentive could be made stronger by modifying the structure and/or purpose of the detention costs.

Keywords: Empty container management; Markov decision process; optimal inventory policy; value iteration technique; time-based threshold.

1 Introduction

Containerization is the trend towards the deployment of a standard and dedicated international transport system for containers from door-to-door. Containerization has shaped global supply chains by providing reliable, low cost and secure service for international trade. Container-based trade has expended from 50 million Twenty-foot Equivalent Units (TEUs) in 1996 to 180 million TEUs in 2015, such that containerized cargo now represents more than half the value of all international seaborne trade (Trade and Development (2016)). As a result, many authors consider ocean container transport as critical for global supply chain performance (Fransoo and Lee, 2013).

Container transport has many advantages including standardization, ease of handling, protection against damage and security. However, empty container movement is considered as an important disadvantage of containerization. Once emptied at destination, the container often needs to be repositioned to be filled in again. For instance, the review of maritime transport published by the United Nations (2011) highlights that the costs of seaborne empty container repositioning was estimated to \$20 billion in 2009, while the costs of empty container repositioning in the hinterland was around \$10 billion for the same year. Globally, empty container repositioning accounted for 19% of the global industry income in 2009 (United Nations, 2011).

In this article, we focus on empty container management in the hinterland of a deep-sea port, i.e., at the regional level (Boile et al., 2008). Consider an import container emptied in the hinterland and assume that this one needs to be reloaded with export cargo. Although inefficient, the common practice consists of having first the empty container being shipped back to the port, before being sent to the shipper. Besides the financial impacts associated with this practice, unnecessary movements of empty containers lead to negative societal impacts such as pollution, congestion and accidents. In this article, we focus on *street-turns*; the most straightforward strategy to reduce the unnecessary movements of empty containers. The idea behind a street-turn is very basic and consists of shipping the empty container directly from the consignee to the shipper without passing by a terminal. There are two important advantages of the street-turn strategy. First, empty movements are reduced and accordingly repositioning costs decrease. For example, Hjortnaes et al. (2017) estimate that street-turns (referred to as direct repositioning in their article) result in cost savings of up to 17%. The reduction of empty movements may also imply a decrease in the number road accidents. Each street-turn also avoids two movements to and from the terminal, reducing congestion. Second, empty container demand from the shippers can be met sooner, increasing the container utilization rate which positively impacts container fleet sizing (Jula et al., 2006; Dong and Song, 2012).

Even if the advantages of the street-turn strategy appear to be important for all parties involved, Lei and Church (2011) highlight that street-turns are only used 10% of the time in the hinterland of Los Angeles/Long Beach. Identically, Wolff et al. (2007) find that the share of street-turns was in a range of 5-10% in the hinterland of the port of Hamburg. Consequently, Braekers et al. (2011) estimate that empty containers account for 40% to 50% of the regional movements. There are multiple explanations for the limited use of street-turns. Among them, we highlight here that consignees are generally not involved in empty container management as containers are generally controlled by the shipping lines who own and/or lease a pool of containers. Therefore, empty containers that becomes available at a consignee's site are shipped back to the shipping line before an export match could be identified. This practice is highlighted by Lee and Song (2017) who state that regional container movements are operated by actors beyond the control of ocean carriers. This article explores the potential for consignees to proactively manage empty containers at their location to enhance the feasibility of street-turns. From our knowledge, this article is the first one to investigate this option. The street-turn strategy is traditionally studied from a shipping line perspective (Jula et al., 2006; Deidda et al., 2008; Furió et al., 2013; Sáinz Bernat et al., 2016).

We focus on a merchant haulage setting as this context is particularly relevant to study the role of consignees in empty container management. Under merchant haulage, the shipping lines charge detention and demurrage fees. Demurrage fees are incurred when the container stays at the deep-sea terminal, and detention fees are charged when the container is in the hinterland, until being shipped back to the shipping line. From the perspective of the shipping company who owns the containers, containers are assets in the hinterland and therefore, they represent some opportunity costs. A shipping line executive clearly highlighted this fact during our preliminary interviews by stating that the container should make money in each single part of the chain. Shipping lines also use detention and demurrage fees as a way to indirectly control the containers in the hinterland in order to ensure timely return of their asset. As a result, detention and demurrage fees are relatively high in practice. However, consignees tend to negotiate detention and demurrage free periods. This leads to a complex structure in practice, with a free period, and several levels of fees. As an example the official detention fees for a 40 ft. dry container imported to the port of Rotterdam and transported further by truck include a free period of 3 days, then the rate is €55/day from day 4 to day 7 and finally the rate is €85/day after 7 days (Maersk, 2016). In our discussions with consignees across Europe (Netherlands, France and Sweden), detention fees are always mentioned as one of the main barriers against street-turn strategies as the consignees feel that they do not have enough time to identify an export match before incurring high detention costs.

In this article, we assess whether consignees might take a proactive role in empty container management by formulating an inventory model for empty container management under merchant haulage. The decision to take for the consignee at any point of time and for any container is either to keep the container for a future street-turn or to send it back to the shipping line. The traditional decision variable for such a problem is the quantity of containers in inventory. This choice is known to be optimal for linear inventory holding costs (Li et al., 2004; Song and Zhang, 2010; Zhang et al., 2014). However, this decision variable is no longer optimal under non-constant detention fees per time unit. The ages of each container in inventory should also be considered. The derivation of the optimal policy based on the number of containers together with their ages is challenging and will most likely lead to a very complex structure. Such a policy is unlikely to be implemented. Instead, based on the data of Maersk (2016) which suggest a strong time-dependency of the detention fees, we only consider the ages of the containers as a decision variable. This choice also follows from the practice where the tracking of containers has significantly improved over the last years.

The contributions of the article are as follows.

- From a technical perspective, we prove with non-constant holding costs that a time-based threshold policy is optimal among the class of time-based policies. Using a value iteration technique, we prove the result on a model where the time spent by the oldest container at the consignee is discretized. To the best of our knowledge, this article is the first to prove the optimality of a threshold structure based on the time spent in the system using the value iteration method. Next, following a Markov chain analysis, we derive closed-form expressions of the performance measures and the optimal cost per container. Based on proven monotonicity properties of the performance measures, we next develop an algorithm to compute efficiently the optimal time-threshold. In addition, an asymptotic analysis is proposed to get closed-form expressions of the threshold in particular regimes.

- From a managerial perspective, using the numerical values of Maersk (2016), we show that the proactive management of empty containers by consignees enables reaching a high level of street-turns. Moreover, we show that the cost saving ranges from 16% to 69% based on the data from the Netherlands. We also identify other conditions for which the incentive is relatively limited. Therefore, we further investigate if the incentive could be made even stronger by increasing the detention free period or by changing the detention fees structure. Without modifying the rewards of the shipping company, these changes may reduce the intensity of empty containers transportation.

The remainder of this paper is structured as follows. We conclude this section with a short literature survey. Section 2 describes the model. Section 3 provides the proof of the optimal policy and a formula for the long-run expected costs. Section 4 exploits these results to study numerically the behavior and performance of the optimal inventory policy on the level of street-turns. A series of practical insights follows from this study. Section 5 gives concluding remarks. All proofs and a reminder of the notations are given in the Appendix at the end of the article.

The literature on containerization is quite extensive. We refer to Levinson (2010) for a historical perspective on containerization. Empty container management has deserved a lot of attention from the transport and maritime economics communities. Many relevant and inspiring results have been generated. However, the transport and maritime economics literature does not take the inventory perspective into account, and hence their research paradigms cannot be used to tackle the problem we consider here. We refer to Dejax and Crainic (1987) for a review of early works of the Operations Management/Transportation Science community in this field and we refer to Braekers et al. (2011), Song and Dong (2015) and Lee and Song (2017) for recent overviews. Lee and Song (2017) classifies the existing contributions into two categories. The first one applies network flow models to the empty container repositioning problem. The second category considers the empty container repositioning problem from an inventory theory perspective by considering empty containers as inventories that enable meeting customer demand. For instance, Lee and Song (2017) highlight that many shipping lines are using inventory-based policies to reposition empty containers. The related contributions provide evidences of the optimality of control policies based on quantity-thresholds and sometimes provide closed-form solutions for these threshold levels. More specifically, Li et al. (2004), Song and Zhang (2010), Zhang et al. (2014) focus on a single empty depot located in a port and controlled by a shipping line. (Song, 2005, 2007), Lam et al. (2007), Shi and Xu (2011), Ng et al. (2012), Xie et al. (2017) focus on empty container (or equivalently vehicle) management for a two-depot system. Finally, Du and Hall (1997), Li et al. (2007), Song and Dong (2008), Yun et al. (2011), Dang et al. (2012) and Dang et al. (2013) focus on more general networks. Considering empty containers from an inventory control perspective also allows to make another link with the existing literature as this setting shares some similarities with remanufacturing/disposal models in the reverse logistics context (see e.g., Heyman (1977) or Teunter and Vlachos (2002)).

2 Problem Formulation

Setting. We consider the regional empty container management problem from the consignee's perspective in isolation from the problem of managing the cargo inside the containers. The inventory management decisions traditionally focus on the cargo and exclude the transportation packaging units. As a first attempt in analyzing the management of empty containers from the consignee perspective, we focus on models that do not affect the management of the cargo itself. The supply of empty containers is consequently not considered as a decision variable in our model and we consider this one as stochastic. We assume that the arrival process of empty containers is Poisson with parameter λ. Once an empty container is made available at the consignee's site, two options are available. Either the empty container is shipped back to the shipping line (either to the deep-sea terminal or to an empty depot depending on the cheapest available option) with a cost of c_{sl} monetary units per empty container, or the consignee identifies an export match and incurs a cost of c_{st} monetary units per empty container. We consequently define the repositioning costs as $c_s = c_{sl} - c_{st}$ and we assume that $c_s > 0$ (otherwise, there is no interest in holding empty containers for the consignee). We further assume that the aggregated demand for empty containers is stochastic and follows an exponential distribution with rate μ. We refer to a single shipper for clarity reasons.

The objective for the consignee is to find the optimal empty containers inventory policy which minimizes the long-run expected costs. In case an empty container is kept in inventory, the consignee incurs linear *holding costs* due to physical storage costs as well as *detention costs*. We aggregate these costs and we refer to them as inventory holding costs for simplicity. We assume that inventory holding costs are increasing and convex in the time spent in inventory (in accordance with the structure of detention fees), hence the oldest container in inventory is also the most costly one. Therefore, it is optimal to apply a first-in-first-out policy for sending back containers (either to the shipping line or for making a street-turn). We denote by $c(t)$ the inventory holding cost of a container which has stayed exactly t time units in the inventory. In summary, the inventory at the consignee is modeled by:

- A Poisson arrival process of empty containers with parameter λ;
- An exponential demand from the shipper with parameter μ;
- A first-in-first-out discipline;
- A cost for sending back containers to the shipping line of c_s per container;
- An increasing and convex inventory holding cost $c(t)$ per container which has stayed exactly t time units at the consignee.

The considered model, depicted in Figure 1, is quite stylized and may not represent the overall complexity of the real system. It however aims to capture its main features which are the stochasticity in arrivals and demands, the repositioning cost and the non-linearity of the holding cost.

Optimization problem. We consider the set of all non-preemptive non-anticipating first-in-first-out policies for sending back containers to the shipping line. At any point of time, we want to decide for the oldest container (if any) whether to keep it, or to send it back to the shipping line. Whenever a match is possible it is optimal to realize the match. The objective function is composed by the inventory holding costs (including detention costs) and the costs of sending back a container to the shipping line. The goal is to find the optimal policy which minimizes the long-run expected cost per container; $E(C)$.

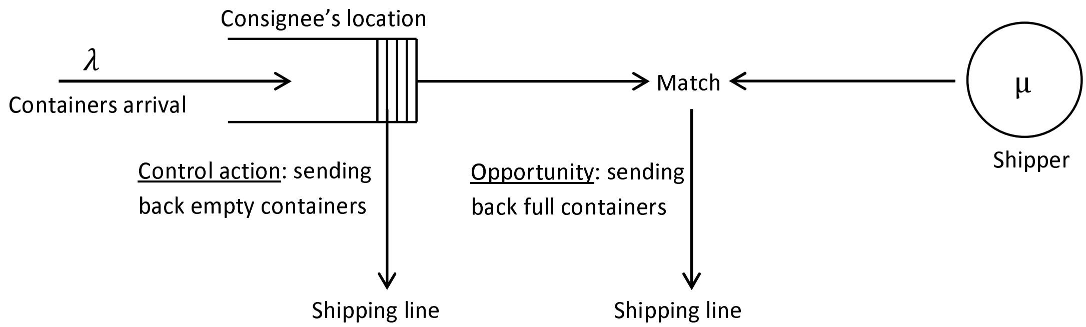


Figure 1: The Model

3 Time-based Threshold Policy Analysis

We first prove in Section 3.1 that the optimal control policy is a threshold policy on the age of the oldest container. Next, in Section 3.2 we compute the performance measures and the optimal cost under this policy. In Section 3.3, we further propose an asymptotic analysis which leads to explicit expressions of the optimal threshold.

3.1 Optimal policy

A state description based on the number of containers in the hinterland is common in Markov chain analysis, it however does not allow to evaluate the overall cost per time unit of a set of containers due to the non-linear inventory holding cost structure. The information on the age of each container should be added in order to completely describe a given state of the system. This would make the state description very complex. Yet, combining the Poisson arrival process with the first-in-first-out discipline, the age of the oldest container allows us to evaluate at any time the distribution of the containers as a function of their age. Therefore, the age of the oldest container is chosen as a decision variable to determine the optimal policy.

We propose to formulate the problem as a Markov decision process and next use the value iteration technique to prove the form of the optimal policy. We use a non-traditional approach for the modeling of the queueing system, as proposed in Koole et al. (2012). The idea is to discretize the waiting time of the Oldest Container at the Consignee (OCC) by a succession of exponential phases with rate γ per phase instead of using the traditional definition of the number of containers in the queue. As shown in Koole et al. (2012), as γ tends to infinity, this approximate setup converges to the original one, which in turn leads to an exact analysis.

Let us denote by x a state of the system, where $x \geq 0$. State $x = 0$ corresponds to an empty inventory. States with $x > 0$ correspond to a situation where the OCC has an age of x phases. Lumping together the states representing the empty system and the time spent by the OCC at the consignee in one dimension can be done as the system cannot be empty while containers are waiting. We next describe the 3 possible transitions. When the OCC changes, because of a shipper's demand (see transition Type 2), the age phase changes from $x > 0$ to $x - h$ with probability $q_{x,x-h}$, where $q_{x,x-h} = \left(\frac{\lambda}{\lambda+\gamma}\right)\left(\frac{\gamma}{\lambda+\gamma}\right)^h$ for $0 \leq h < x$ and

$q_{x,0} = \left(\frac{\gamma}{\lambda+\gamma}\right)^x$ (Koole et al., 2012).

1. An arrival with rate λ while the system is empty ($x = 0$), which changes the state to $x = 1$, then the OCC entity is created.

2. A shipper's demand with rate rate $\mu q_{x,x-h}$ while the system is not empty ($x > 0$), which changes the state to $x - h$, that is, the new OCC is in waiting phase $x - h$.

3. A phase increase with rate γ while the system is not empty ($x > 0$), which changes the state to $x + 1$. The waiting phase of the OCC is increased by 1.

Let us denote by $V_n(x)$ the expected cost over n steps, for $n \geq 0$ and $x \geq 0$. We pay a cost of c_s per container sent back to the shipping line and let $c(x)$ be the inventory holding cost function per container defined for $x > 0$. We assume that $c(x)$ is a increasing and convex function in x. The cost $c(x)$ can therefore be used to model both the physical storage cost and the detention cost which may include a Detention Free Period (DFP). Since the total event rate is bounded, our continuous-time model is uniformizable (Section 11.5.2. in Puterman (1994)). This allows us to consider the system only at transition instants and simplify the problem formulation. The uniformization is done using the maximal event rate $\lambda + \mu + \gamma$, that we assume equal to 1. We denote by F the operator on the set of functions f from $\mathbb{N}$ to $\mathbb{R}$ defined by $F(f(x)) = \sum_{h=0}^{x} q_{x,x-h} f(x-h)$ for $x > 0$, and $F(f(0)) = f(0)$ for $x = 0$. For $n \geq 0$, we have

$$\begin{aligned} V_{n+1}(0) &= \lambda W_n(0) + (1-\lambda)V_n(0), \\ V_{n+1}(x) &= \gamma W_n(x) + \mu(F(V_n(x)) + c(x)) + (1-\gamma-\mu)V_n(x), \text{ for } x > 0, \end{aligned}$$

with
$W_n(x) = \min(F(V_n(x)) + c_s + c(x), V_n(x+1))$ if $x \geq 0$. We assume that $V_0 = W_0 = 0$.

For each $n > 0$ and every state x ($x \geq 0$) there is a minimizing action: send an empty container to the shipping line or keep all containers in the inventory. For fixed n ($n \geq 0$) we call this function:

$$\mathbb{N} \to \{\text{keep}, \text{send}\},$$

a policy. As n tends to infinity, this policy converges to the average optimal policy with the convergence of $V_{n+1} - V_n$, that is, the policy that minimizes the long-run expected average costs. The convergence results from the aperiodic irreducible countable-state Markov chains considered here. The aperiodicity is due to the fictitious transitions from a state to itself. Then, Theorem 8.4.5 of (Puterman, 1994) guarantees the existence of an optimal deterministic stationary policy.

In Theorem 1, we prove by induction that the optimal policy is of threshold type under the condition $\gamma \geq \lambda$. More precisely, all containers are allowed to join the inventory, regardless of the system state. However, the system does not allow containers to infinitely stay in the inventory. There exists a time limit, referred to as the Acceptable Detention Time (ADT), such that if a container has waited since exactly ADT time units, then it is sent back to the shipping line. Since our discretized model converges to the continuous one as γ tends to infinity, the condition $\gamma > \lambda$ is automatically satisfied for the exact model.

Theorem 1 *For $\gamma > \lambda$, the optimal policy for sending back containers at iteration n is of threshold type. There exists a threshold on the number of phases at which an empty container is sent back to the shipping line.*

3.2 Performance analysis

We have proven in Section 3.1 that the optimal policy for the consignee is a threshold policy on the age of the empty containers. We evaluate here the performance of the system under this policy using the approach developed in Legros (2016). We approximate the deterministic duration before being sent back to the shipping line by an Erlang random variable with n phases and rate γ per phase. We choose n and γ such that $\frac{n}{\gamma} \triangleq \text{ADT}$. The Laplace transform of the Erlang distribution with parameters n and γ is $\left(\frac{\gamma}{\gamma+s}\right)^n$. We have

$$\left(\frac{\gamma}{\gamma+s}\right)^n = e^{n\ln((1+s/\gamma)^{-1})} \underset{\gamma\to\infty}{\sim} e^{n\ln(1-s/\gamma)} \underset{\gamma\to\infty}{\sim} e^{-ns/\gamma} = e^{-s\text{ADT}},$$

where we write $f(a) \underset{a\to a_0}{\sim} g(a)$ to express that $\lim\limits_{a\to a_0} \frac{f(a)}{g(a)} = 1$, for $a_0 \in \mathbb{R}$. Applying the Levy continuity theorem for Laplace transforms, this result ensures that as n and γ go to infinity, the considered Erlang random variable converges in distribution to the deterministic duration before being sent back to the shipping line, ADT.

We use the same state definition as in Section 3.1 except that the total number of phases is bounded by n. The transition structure is identical to the one in Section 3.1. The only difference is the transition from state $x = n$; a transition from n to a state $n - h$ can be caused not only by a μ-transition but also by a γ-transition which represents then a container sent back to the shipping line. In Theorem 2, we give closed-form expressions for the probability of an empty system p_0, the proportion of containers sent back to the shipping line, P_s, the expected time spent by a container in the inventory, $E(T)$, the expected number of container in the inventory, $E(N)$, and the proportion of empty containers which has spent more than t in the inventory, $P(T > t)$ with $0 \leq t \leq \text{ADT}$. These expressions are given as functions of the ratio $a = \lambda/\mu$. The ratio a represents the import/export balance.

Theorem 2 *We have*

$$\begin{aligned} p_0 &= \frac{1-a}{1-ae^{-ADT(\mu-\lambda)}}, \qquad P_s = p_0 \cdot e^{-ADT(\mu-\lambda)} \\ E(N) &= a\frac{1-e^{-ADT(\mu-\lambda)}(1+aADT(\mu-\lambda))}{(1-a)(1-ae^{-ADT(\mu-\lambda)})}, \qquad E(T) = \frac{E(N)}{\lambda}, \text{ and} \\ P(T>t) &= \mathbb{1}_{t<ADT}\frac{e^{-t(\mu-\lambda)} - ae^{-ADT(\mu-\lambda)}}{1-ae^{-ADT(\mu-\lambda)}}, \end{aligned}$$

where $\mathbb{1}_{(x\in A)}$ is the indicator function of a subset A.

Note that our system reduces to an M/M/1+D queue where the arrival process is generated by the arrival of empty containers and the service is ensured by the shipper. The metrics provided above were already derived. We refer the reader to Graves (1978) for the derivation of the first four metrics in a context of perishable inventory and to Baccelli and Hebuterne (1981) for the derivation of the waiting time distribution in an M/M/1+D queue.

Using the expression of $P(T > t)$, one may obtain the probability density function of the time spent in the system by a matched container; $\frac{\partial(1-P(T>t))}{\partial t}$. This leads to the expression of the expected cost per container in Corollary 1.

Corollary 1 *We have*

$$E(C) = (c_s + c(ADT)) \cdot \frac{(1-a)e^{-ADT(\mu-\lambda)}}{1-ae^{-ADT(\mu-\lambda)}} + \int_{t=0}^{ADT} \mu \frac{(1-a)e^{-t(\mu-\lambda)}}{1-ae^{-ADT(\mu-\lambda)}} \cdot c(t)\,\mathrm{d}t. \tag{1}$$

In Proposition 1, we give the first and second order monotonicty properties of the main performance measures as a function of the control parameter ADT. These properties will be used to develop a method to compute numerically the optimal threshold for a general cost function. Moreover, these results will also be used in Section 4 to better explain the behavior of the expected cost.

Proposition 1 *For $a > 0$ and $ADT > 0$, the following holds:*

1. *The proportion of containers sent back to the shipping line, P_s, is strictly decreasing and strictly convex in ADT (the proportion of matches is thus strictly increasing and strictly concave in ADT).*

2. *The expected time spent at the consignee $E(T)$ and the expected number of containers are strictly increasing in ADT.*

3. *The proportion $P(T > t)$ is strictly increasing in ADT for $t < ADT$.*

4. *Given that $c(t)$ is a continuous piecewise linear increasing in t, the expected cost per container, $E(C)$, is first decreasing and next increasing in ADT.*

Note that the expected time spent at the consignee as well as the expected number of containers at the consignee are neither convex nor concave in ADT for ADT > 0. Note also that these monotonicity results apply for the M/M/1+D queue. The last statement of the proposition allows us to propose the following simple algorithm to obtain the optimal threshold.

Algorithm: Computation of the optimal threshold. We denote by c_k, the slope of the function $c(t)$ on the interval $[t_{k-1}, t_k)$ with the convention that $t_0 = 0$ for $k \geq 1$.

`Start with` $k = 1$. `Go to line 1.`

1. `If` $c_s\mu > c_k$`, solve`

$$c_s\mu(1-a)^2 = c_k(1 - 2a + a(1-a)\mu\mathtt{ADT} + a^2 e^{-\mu(1-a)\mathtt{ADT}}). \tag{2}$$

   `Denote by` $\mathtt{ADT}_k$ `the solution of this equation. If` $\mathtt{ADT}_k \geq t_k$`, increase` k `by 1 and go back to line 1. If` $t_{k-1} \leq \mathtt{ADT}_k < t_k$`, go to line 2. If` $\mathtt{ADT}_k < t_{k-1}$`, go to line 3.`
   `If` $c_s\mu \leq c_k$`, go to line 3.`

2. `The optimal solution is` $\mathtt{ADT}_k$`.`

3. The optimal solution is t_{k-1}.

Corollary 2 *On each interval $[t_{k-1}, t_k)$, for $k > 0$, the optimal threshold ADT is increasing and piecewise concave in c_s and decreasing and piecewise convex in c_k. Moreover, the proportion of matches, the expected number of empty containers at the consignee and the optimal cost per container are increasing and piecewise concave in c_s and decreasing and piecewise convex in c_k.*

3.3 Asymptotic analysis

Even if Equation (2) in the algorithm can be easily solved numerically, it does not lead to an explicit expression of the optimal threshold. We therefore additionally provide explicit expressions of the optimal threshold in extreme cases of the import/export balance. These expressions are derived from Equation (2) using Taylor expansions and equivalent expressions. The following holds

- As a tends to infinity (if the import/export balance is high), Equation (2) leads to

$$\text{ADT} \underset{a\to\infty}{\sim} \frac{1}{\lambda}\ln\left(\frac{c_s\mu}{c_k}\right).$$

- As a tends to zero (if the arrival of containers is low or if the demand is high), Equation (2) leads to

$$\text{ADT} \underset{a\to 0}{\sim} \frac{1}{\lambda}\left(\frac{c_s\mu}{c_k} - 1\right).$$

In Table 1, we evaluate the expected cost obtained using these two expressions of the threshold in comparison with the use of the optimal threshold for different values of the arrival rate in the case of a constants holding cost per time unit, c_1. In the second and the third column we give the optimal threshold and its related expected cost. We compute the relative difference between the expected cost obtained with the two approximations and the optimal expected cost by rd= $\frac{E(C)_{\text{approximation}} - E(C)_{\text{optimal}}}{E(C)_{\text{optimal}}}$. As expected, Table 1 reveals that the first approximation is the best for high arrival rate situations whereas the second one is the best for low arrival rate situations. However, it is interesting to observe that under low arrival rate situations the first approximation still performs well. The explanation is given by the result $\ln(1+x) \underset{x\to 0}{\sim} x$.

Table 1: Performance comparison ($\mu = 1$, $c_s = 80$, $c_1 = 5$)

λ	Optimal policy ADT$_{opt}$	$E(C)_{opt}$	Approximation 1 ADT$_1 = \frac{1}{\lambda}\ln\left(\frac{c_s\mu}{c_u}\right)$	$E(C)_1$	rd$_1$	Approximation 2 ADT$_2 = \frac{1}{\lambda}\left(\frac{c_s\mu}{c_u} - 1\right)$	$E(C)_2$	rd$_2$
0.01	1485.00	0.05	277.26	0.05	0.00%	1500.00	0.05	0.00%
0.1	135.10	0.56	27.73	0.56	0.00%	150.00	0.56	0.00%
0.25	45.35	1.67	11.09	1.67	0.15%	60.00	1.67	0.00%
0.5	16.00	5.00	5.55	5.68	13.66%	30.00	5.00	0.02%
0.75	7.55	13.29	3.70	15.48	16.46%	20.00	14.70	10.54%
1	4.57	27.84	2.77	29.97	7.67%	15.00	44.84	61.08%
1.5	2.51	63.80	1.85	65.04	1.94%	10.00	105.54	65.42%
2	1.73	102.25	1.39	102.99	0.72%	7.50	150.04	46.74%
3	1.06	180.94	0.92	181.27	0.18%	5.00	232.50	28.50%
5	0.60	340.03	0.55	340.15	0.04%	3.00	393.75	15.80%
10	0.29	739.42	0.28	739.45	0.00%	1.50	794.44	7.44%
100	0.03	7938.92	0.03	7938.92	0.00%	0.15	7994.95	0.71%

Hence, the second expression found in the case where a tends to zero is equivalent to the first one found in the case where a tends to infinity. Note also that the worst degradation of the performance is observed with the second approximation.

4 Implementation of the TBP; Analysis and Insights

We conduct here a numerical analysis based on the values of Maersk (2016) to derive the key findings of this study related to the implementation of the TBP.

Setting. Recall from Section 1 that the official detention fees according to Maersk (2016) include a DFP of 3 days, then the rate is €55/day from day 4 to day 7 and finally the rate is €85/day after 7 days. In addition, we assume that the physical storage cost is equal to €5/day. Thus, we have $c_1 = 5$, $c_2 = 60$ and $c_3 = 90$. This allows us to specify the definition of the cost function:

$$c(t) = \begin{cases} 5t & \text{if } 0 \leq t \leq 3, \\ 60t - 165 & \text{if } 3 \leq t \leq 7, \\ 90t - 375 & \text{if } t \geq 7. \end{cases}$$

We illustrate the optimal policy as a function of c_s for different values of the arrival rate and an expected demand, μ, of 1 container per day. We take $a = 1$, $a = 2$ and $a = 5$ as examples. These values are representative of imbalance in major trade lanes. For instance, Theofanis and Boile (2009) report that trade imbalance for the transpacific lane increased from 50% (i.e., $a = 2$) to 67% (i.e., $a = 3$) from 2000 to 2005. In addition, we compare our optimal policy with the common practice that consists of sending back directly all containers, referred to as the *immediate return policy* (IRP).

4.1 Values of the TBP

We investigate here the potential that the implementation of the TBP may have on street-turns and on the control of the containers' inventory. We also analyze the impact on costs that this implementation may have in import zones.

4.1.1 A high potential of street-turns

In Figure 2, we evaluate the proportion of matches under the optimal policy as a function of c_s. This proportion is computed as the ratio of the expected number of matches and the expected demand from the shipper on a given period of time. We consider cases with $\lambda \geq \mu$ so the maximum number of matches is limited by the demand for empty containers by the shipper. As proven in Corollary 2, the proportion of matches is increasing and piecewise concave in c_s. Moreover, we observe that the proportion of matches is very high as soon as $c_s\mu > c_1$. This means that empty container management by consignees shows a very high potential for reducing unnecessary movements of empty containers in the hinterland. For reference, within the Netherlands, the cost c_s is about €100. This leads to a proportion of matches exceeding 70% even for $\lambda = 1$. We thus get the following insight:

Insight 1 *The proactive management of empty containers by the consignees enables reaching a high level of*

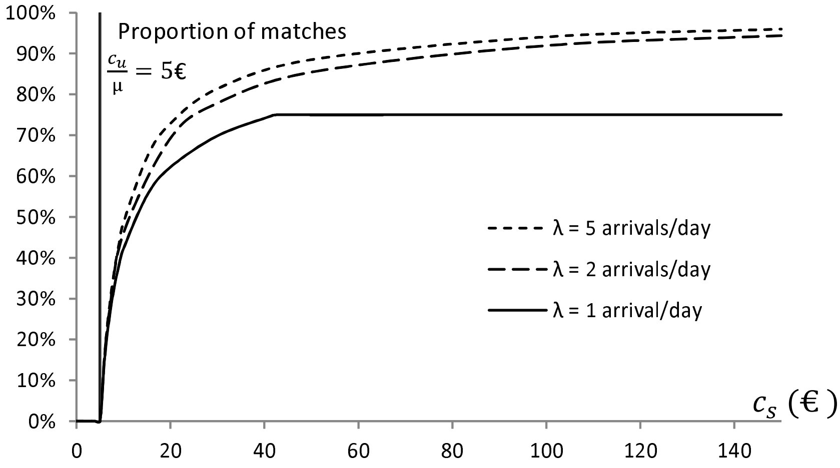


Figure 2: Proportion of matches

street-turns in import zones.

Street-turns are beneficial not only for consignees (as a way to decrease their costs) but also for shipping lines (by increasing the utilization rate of the containers and reducing the cost of sending back empty containers to shippers). Additionally, the literature on street-turns highlighted in Section 1 shows that street-turns enable reducing congestion, accidents and pollution. Insight 1 highlights that the new management practice studied in this article may strongly help in optimizing container flows in the hinterland.

4.1.2 A good control of the expected number of containers

The TBP may be criticized by its indirect control of the quantity in the inventory. This critic is particularly relevant if the consignee's capacity is limited and if the variations in the arrival rate are important. Thus, in Figure 3 we evaluate the optimal expected number of containers in the inventory. As proven in Corollary

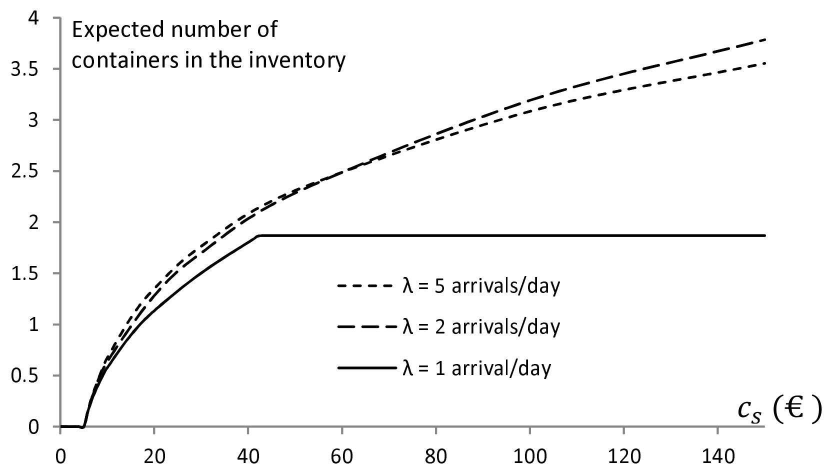


Figure 3: Expected number of containers

2, the expected number of container in inventory is increasing and piecewise concave in c_s. Moreover, it is interesting to observe that the sensitivity of the expected number of containers is decreasing in λ. This means that for high arrival rates, the variation of the arrival rates have a little impact on the number of containers at the consignee. This leads to the following insight:

Insight 2 *In import zones, the variations of the arrival rate have a little impact on expected number of containers at the consignee.*

Note however that this affirmation is only true in expectation. We can additionally notice that the curves for $\lambda = 5$ and $\lambda = 2$ cross each other. So, if c_s is low the size of the inventory increases with λ and the opposite holds when c_s is high. For low values of c_s, the number of containers in inventory increases in λ since more arrivals allows the consignee to store more while neglecting the eventual repositioning costs. For high values of c_s, the objective is to maximize the proportion of matches in order to reduce as much as possible the flow of repositioned containers. So, for low arrival rates, more containers at the location are needed to answer the demand.

4.1.3 An impact on costs that strongly depends on the setting

In Figure 4, we compute the optimal cost per container. As stated in the algorithm of Section 3.2, if $c_s\mu \leq c_1$,

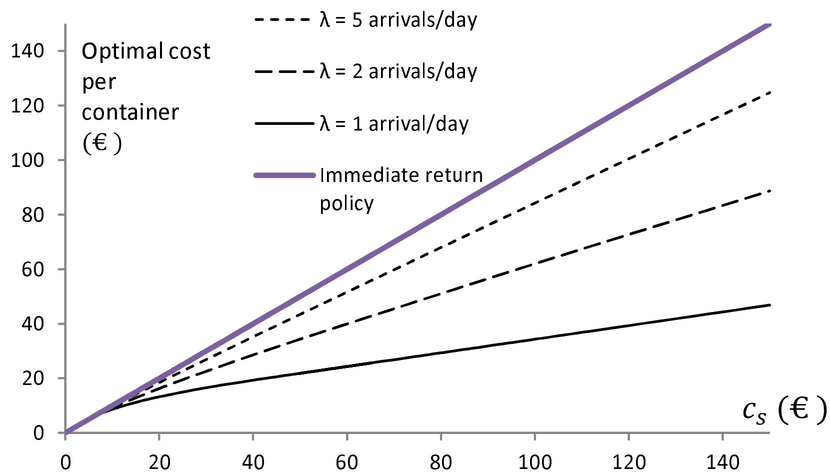


Figure 4: Optimal cost per container

then the IRP is optimal, i.e., empty container management is not interesting for the consignee. Otherwise, as c_s increases, the absolute and relative profit obtained when applying the optimal policy (as compared to the IRP) increases. As mentioned above, within the Netherlands, the cost c_s is about €100, which implies that the potential savings per container are from 16% for $\lambda = 5$ to 69% for $\lambda = 1$. This shows that the policy we propose can help consignees to significantly reduce their inbound transportation costs. However, if c_s is small, we can notice that the cost saving is strongly reduced. For instance, if $c_s =$€30, the savings per container is only 8% for $\lambda = 5$. Note also that, the benefit of implementing the optimal policy decreases as the arrival rate of containers increases. Recall that the proportion of matches increases as the arrival rate of empty containers increases. This means that the incentive to manage empties is lower in the case this practice could be the most impactful. This leads to a third insight:

Insight 3 *In import zones, empty container management by the consignee can significantly reduce inbound transportation costs. However, the savings are lower under the most favorable conditions for street-turns, i.e., in case of strong imbalance.*

This third insight may be quite disappointing at first glance as we have highlighted that many hinterlands are subject to strong trade imbalance. Combined with Insight 1, we face a very effective management practice to handle the problem of reducing empty movements of containers in the hinterland, but we show that this practice may not necessarily be applied under the conditions that are the most favorable for street-turns.

4.2 Incentives for managing containers at location

The results highlighted above show that containers' management by consignees leads to cost savings that are strongly dependent on the conditions faced by consignees. Therefore, we investigate here if the incentives could be made stronger. This leads us to consider the possibilities of implementing linear detention cost or extending the duration of the DFP. These remedies could be used against the negative effect that detention fees have on the willingness consignees have to implement empty container management policies.

4.2.1 On the usefullness of detention fees

In Figure 5, we compute the optimal threshold. As proven in Corollary 2, the optimal threshold is increasing and piecewise concave in c_s and it never exceeds the DFP. For instance with $\lambda = 1$, we need to have $c_s > 654$ to obtain an optimal threshold exceeding the DFP. This corresponds to the behavior of the consignees we

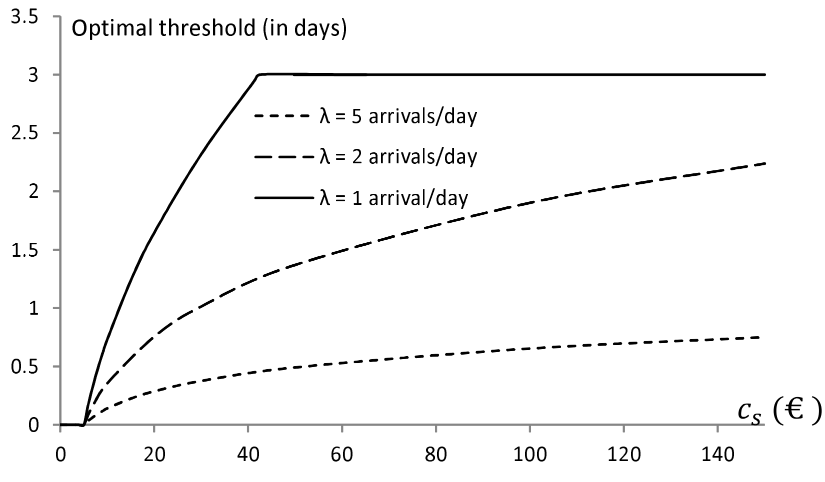


Figure 5: Optimal threshold

interacted with in several European countries as they highlight that they always try to avoid paying detention fees. This may lead us to conclude that detentions fees are efficient in controlling the time spent by containers in the hinterland, in line with one of the main purpose of these fees.

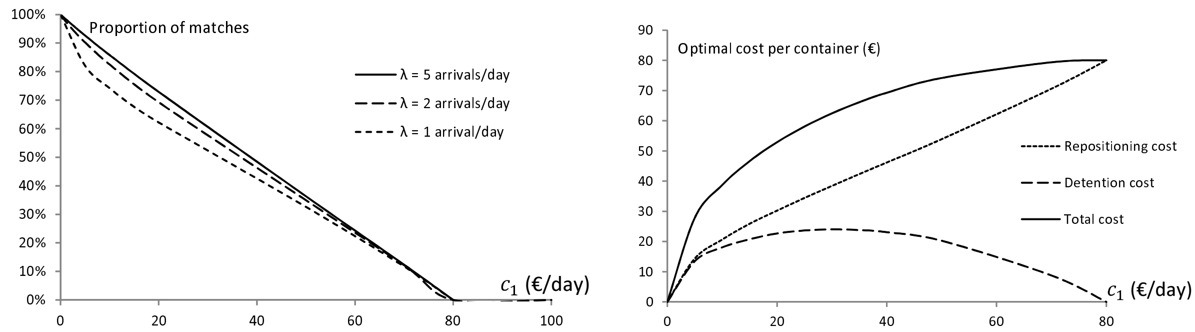
Assume that we omit the potential revenue that shipping lines obtain from their containers in the hinterland and assume that we omit detention fees. Many shipping line companies would be reluctant to implement such type of tariff as they would be afraid of losing control of their containers, by incentivizing consignees to keep empties for a long time. However, we can notice from Figure 5 that the proposed optimal policy often leads to send back empties to the shipping line before the end of the DFP. We conclude that empty containers will be shipped back to the shippers quite quickly, independently of detention fees as the physical costs of keeping empties, even if they are small, are sufficient to ensure a low value for the optimal threshold. We need to acknowledge here that we assume that consignees unload their containers directly after arrival, i.e., they do not use containers as cheap storage locations. From our knowledge, this practice is quite uncommon in Europe and North America. However, our study does not enable us to assess if detention fees really deter consignees to do so. We exclude this practice from the analysis in what follows and we derive the following insight.

Insight 4 *Detention fees are not necessary to ensure that containers do not spend too much time in the hinterland. The physical cost of storing empties is enough to deter consignees to keep empties for a long time before shipping them back to the shipping line.*

Insight 4 questions the real objective of the detention fees. Detention fees are often claimed to help shipping lines to control their containers in the hinterland, but they may also be a source of additional revenue for shipping lines. In what follows, we further investigate two scenarios.

4.2.2 Proposing linear detention fee structure

First, we show how the detention fee structure can be modified to encourage consignees to manage empty containers at their location, while protecting revenues generated from detention fees. We investigate the option of proposing a linear detention fee structure, by changing the aim of these fees. Detention fees are nowadays mainly perceived by shippers as *penalty* in case of late delivery of empty containers to the shipping line. We could have them to be perceived as *renting fees*, for the equipment shipping lines provide to the consignee in case of merchant haulage. Assume, for instance, that the shipping line proposes a new detention tariff which consists of a single rate of €15/day (independently of the time spent by the container in the hinterland). Assume that this rate has been estimated to generate the same revenue as under the complex detention fee structure exposed in Section 1. This leads to a linear inventory holding costs c_1 of €20 in our model (€15 detention fee + €5 physical storage costs). The results appear in Figure 6. Figure 6(a) shows that the proportion of match is high, so this linear detention fee structure can be helpful in improving the direct reuse of containers in the hinterland. Of course, this may not work if the rate chosen is too high, but we expect that the linear rates proposed by shipping lines would be low if they aim at equalizing with the revenues currently generated. To better investigate this statement, we illustrate a decomposition of the costs incurred by the consignee in Figure 6(b). As proven in Corollary 2, the detention cost per container is concave in the linear detention rate, c_1. So, when increasing c_1, more revenue is generated per container per time unit but the containers stay at the consignee for a shorter period of time. This analysis enables us to highlight that there exists a rate that maximizes the profit from detention fees for the shipping line. This



(a) Proportion of matches

(b) Decomposition of the costs

Figure 6: Optimal policy analysis ($\lambda = \mu = 1$ arrival/day, c_s =€80/container)

rate is equal to €25/day in Figure 6. For this detention rate, the proportion of matches is higher than 50% according to Figure 6(a). This enables us to derive the following insight.

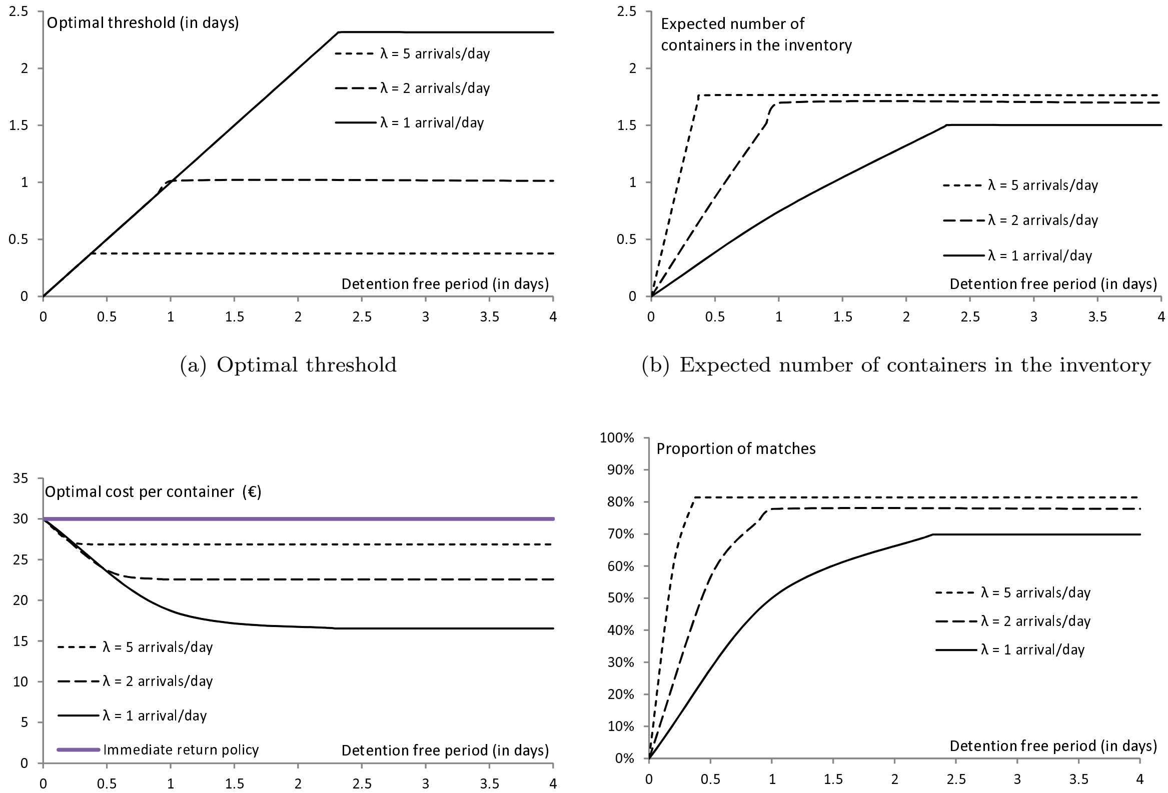
Insight 5 *Linear detention fees may help increasing the proportion of street-turns compared to the current cost structure.*

Linear detention fees would at first be difficult to implement as consignees are used to DFP, but we expect that relabeling them into renting fees would help. Overall, our results show that the detention fee structure used by shipping lines may be overly complex as compared to the targeted result which is primarily the control of containers.

4.2.3 Extending the duration of the DFP

In Figure 7, we present the impact of the duration of the DFP. We take c_s =€30/container and $\mu = 1$ container/day through this example. We vary the DFP from 0 to 4 days. We then consider an inventory holding costs of €60/day (€5 of physical storage costs + Maersk (2016) detention costs) from the end of the DFP to day 7 and finally the rate is €90/day after 7 days (€5 of physical storage costs + Maersk (2016) detention costs). We observe that empty container management by the consignees has a very high potential for reducing the unnecessary movements of empty containers in the hinterland as soon as the residual DFP at the consignee is high enough (Figure 7(d)). This enables us to investigate the following option. The shipping lines could propose to increase the DFP for consignees who accept to hold few empties at their location to proceed to street-turns. This solution will not strongly affect the time spent by containers in the hinterland (as shown in Insight 4) and may be perceived by consignees as a strong incentive to proactively manage empties at their location. Moreover, the TBP will enable consignees to make sure not to keep empties for too long. Also note that this solution may help shipping lines to reduce their repositioning costs incurred when sending empty containers to the shippers and may also help shipping lines to increase the time spent by containers generating revenue in deep-sea vessels. Especially, Figure 7(d) shows that in case the trade imbalance is not very high, an increase in the DFP will have a strong effect on the level on street-turns.

Insight 6 *In order to increase the proportion of street turns in the hinterland, shipping lines may propose an increase in the detention free period for consignees who accept to hold empties at their location.*



(a) Optimal threshold (b) Expected number of containers in the inventory

(c) Optimal cost per container (d) Proportion of matches

Figure 7: Optimal policy analysis

Insight 6 may help shipping lines to identify commercial solutions to entice consignees to proactively manage empty containers. This practice is already in place to entice consignees to use intermodal transportation in the hinterland. As an example, we stated in Section 1 that the DFP for a 40ft. dry container imported to the port of Rotterdam and transported by truck was 3 days. Maersk (2016) additionally states that this DFP is extended to 5 days in case of barge or train transportation. The same type of agreement may therefore be put in place to entice consignees to manage empties at their location.

Conclusion. We summarize here the main results developed in Section 4. Those results, particularly relevant for import zones are that (i) the time-based policy has a high potential of street-turns, (ii) it allows the consignee a good control of the number of containers at location, (iii) cost savings for the consignee are strongly dependent on the conditions. In addition, this study allows us to highlight that the detention fees often do not impact the optimal control since consignees may always choose to send back their containers before the end of the DFP. This leads us to investigate the possibility for shipping lines to implement linear detention fees or to extend the duration of the DFP. In both cases, we find that such decisions may strongly increase the proportion of street-turns.

5 Conclusion

Street-turns are considered as one of the most efficient strategies for empty container management in the hinterland. We investigate if the proactive management of empty containers by consignees could improve the proportion of street-turns in the hinterland. For this purpose, we propose a model of empty container management at the consignee's location. We formulate the problem as a Markov decision process using the age of the oldest container as a decision variable. Next, we prove that the optimal policy is a time-based threshold policy using a value iteration technique. More precisely, all containers are allowed to join the inventory, regardless of the system state. However, a container will be sent back to the shipping line if its waiting time has reached a time threshold without being matched.

We next derive explicit expressions for the performance measures under the optimal policy. This allows us to compute the optimal threshold and to evaluate explicit expressions of the threshold in asymptotic regimes. From a level of street-turns of around 10% reported in the literature, our results show that much higher levels could be achieved if the consignees were proactive in managing empty containers. However, the difference between the total costs per container incurred by the consignee as compared to the costs incurred under the IRP is low. This practice also enables the consignee to reduce container repositiong costs but the incentive varies a lot from one setting to another. So, we further explore if the incentive could be made stronger by modifying the structure and/or purpose of the detention costs. These are preliminary ideas to solve this issue but many other ones may be investigated, such as the sharing of transportation costs between the consignee and the shipper in case of a street-turn. Additionally, other actors in the hinterland such as terminal operators, port authorities and local policy makers may entice consignees to investigate this option. We believe that this article will draw attention on empty container management by the consignees. This may be one of the most powerful and simple option for tackling the problem of empty container repositioning in the hinterland.

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Appendix

Notations

Table 2: Notations

	Exogenous parameters
λ	Arrival rate of empty containers
μ	Demand rate from the shipper
$a = \lambda/\mu$	Ratio between arrival and demand
c_s	Repositioning cost per container
$c(t)$	Inventory holding cost of a container which has stayed exactly t time units in the inventory
c_k	Slope of the function $c(t)$ on the interval $[t_{k-1}, t_k)$ for $k \geq 1$
	Acronym
DFP	Detention Free Period
IRP	Immediate Return Policy
TBP	Time-Based threshold Policy
	Control parameter
ADT	Acceptable Detention Time (control of the TBP)
	Performance measures
p_0	Probability of an empty system
P_s	Proportion of containers sent back to the shipping line
$E(T)$	Expected time spent at the consignee
$E(N)$	Expected number of containers in the inventory
$P(T > t)$	Probability that a container spends more than t time units at the consignee
$E(C)$	Expected cost per container given by Equation (1)

Proof of Theorem 1

We prove by induction that the optimal policy for sending back containers is of threshold type. We thus need to show that $V_n(x+1) - F(V_n(x)) - c_s - c(x)$ is increasing in x, for $x \geq 0$ and $n \geq 0$. In other words, we need to show that

$$F(V_n(x)) + V_n(x+2) + c(x) \geq F(V_n(x+1)) + V_n(x+1) + c(x+1), \tag{3}$$

for $x \geq 0$ and $n \geq 0$. This relation is refereed to as *generalized convexity* (*gcv*). In this proof, we also have to show that V_n is increasing in x, for $x \geq 0$ and $n \geq 0$. Since $V_0(x) = 0$, then V_0 is increasing and generally convex (*igcv*). Note that the arguments $c(x)$ increasing and convex and $c(0) = 0$ are essential to show this property. Although $c(0) = 0$, we still write $c(0)$ in the developed expressions to better show the use of the convex property of c when it is needed.

First, we assume that V_n is *igcv* for a given $n \geq 0$, and we want to show that the same property holds for W_n.

V_n increasing in $x \implies W_n$ increasing in x. First, we show that if V_n is increasing in x, then $F(V_n)$ is also increasing in x. To simplify the notations, we denote by u the ratio $\frac{\lambda}{\lambda+\gamma}$. We have for $x \geq 0$,

$$F(V_n(x+1)) - F(V_n(x)) = \sum_{h=0}^{x+1} q_{x+1,x+1-h} V_n(x+1-h) - \sum_{h=0}^{x} q_{x,x-h} V_n(x-h)$$
$$= \sum_{h=0}^{x-1} u(1-u)^h (V_n(x+1-h) - V_n(x-h)) - (1-u)^x V_n(0) + (1-u)^{x+1} V_n(0) + u(1-u)^x V_n(1)$$
$$= \sum_{h=0}^{x-1} u(1-u)^h (V_n(x+1-h) - V_n(x-h)) + u(1-u)^x (V_n(1) - V_n(0)) \geq 0.$$

Therefore $F(V_n(x))$ is increasing in x.
Using the definition of W_n, one may write,

$$W_n(x) \leq V_n(x+1), \text{ and } W_n(x) \leq F(V_n(x)) + c_s + c(x). \tag{4}$$

If $W_n(x+1) = V_n(x+2)$, then the first inequality in (4) proves that W_n is increasing. If $W_n(x+1) = F(V_n(x+1)) + c_s + c(x+1)$, then the second inequality in (4) proves that W_n is increasing because $F(V_n(x)) \leq F(V_n(x+1))$ and $c(x) \leq c(x+1)$.

V_n is *gcv* $\Longrightarrow$ W_n is *gcv*. We prove the convexity property of W_n. Using the definition of W_n and $q_{x,x-h} = q_{x+1,x+1-h}$, for $0 \leq h < x$, we may write

$$\begin{aligned} W_n(x+1) + F(W_n(x+1)) + c(x+1) &\leq c(x+1) + V_n(x+2) + \sum_{h=0}^{x+1} q_{x+1,x+1-h} V_n(x+2-h) \\ &= c(x+1) + V_n(x+2) + F(V_n(x+2)) + \left(\frac{\gamma}{\gamma+\lambda}\right)^{x+2} (V_n(1) - V_n(0)), \end{aligned} \tag{5}$$

and,

$$\begin{aligned} W_n(x+1) + F(W_n(x+1)) + c(x+1) &\leq c(x+1) + F(V_n(x+1)) + c_s + c(x+1) \\ &+ \sum_{h=0}^{k} q_{x+1,x+1-h} V_n(x+2-h) + \sum_{h=k+1}^{x+1} q_{x+1,x+1-h} \left(F(V_n(x+1-h)) + c_s + c(x+1-h)\right), \end{aligned} \tag{6}$$

for $0 \leq k \leq x$. The right hand side in Equation (5) corresponds to a case where containers should not be sent back if their age is lower than or equal to $x+1$. The right hand side in Equation (6) corresponds to a case where containers should be sent back if and only if their age is strictly higher than k, for $0 \leq k \leq x$.

We distinguish two cases.

- **Case 1: No rejection.** $W_n(h) = V_n(h+1)$ for $0 \leq h \leq x+2$ and $x \geq 0$.

$$\begin{aligned} W_n(x+2) + F(W_n(x)) + c(x) &= V_n(x+3) + \sum_{h=0}^{x} q_{x,x-h} V_n(x+1-h) + c(x) \\ &= V_n(x+3) + F(V_n(x+1)) + \left(\frac{\gamma}{\gamma+\lambda}\right)^{x+1} (V_n(1) - V_n(0)) + c(x). \end{aligned}$$

Note that the second equality is a consequence of $q_{x,x-h} = q_{x+1,x+1-h}$, for $0 \leq h < x$. Since V_n is *gcv*, we have

$$V_n(x+3)+F(V_n(x+1))+c(x+1)\geq V_n(x+2)+F(V_n(x+2))+c(x+2).$$

Therefore,

$$\begin{aligned}
&W_n(x+2)+F(W_n(x))+c(x)-W_n(x+1)-F(W_n(x+1))-c(x+1)\\
&\qquad\geq V_n(x+3)+F(V_n(x+1))+\left(\frac{\gamma}{\gamma+\lambda}\right)^{x+1}(V_n(1)-V_n(0))+c(x)\\
&\qquad\quad -c(x+1)-V_n(x+2)-F(V_n(x+2))-\left(\frac{\gamma}{\gamma+\lambda}\right)^{x+2}(V_n(1)-V_n(0))\\
&\qquad\geq V_n(x+3)+F(V_n(x+1))-V_n(x+2)-F(V_n(x+2))+c(x+1)-c(x+2)\\
&\qquad\quad +c(x+2)+c(x)-2c(x+1)+u(1-u)^{x+1}(V_n(1)-V_n(0))\geq 0,
\end{aligned}$$

because V_n is gcv, $c(x)$ is convex in x and V_n is increasing in x. This proves that W_n is also gcv in this case.

• **Case 2: Rejection above a given waiting phase.** Assume that for a given k ($0\leq k\leq x+1$), we have $W_n(h)=F(V_n(h))+c_s+c(h)$ for $k<h\leq x+2$ and $W_n(h)=V_n(h+1)$ for $0\leq h\leq k$.

$$\begin{aligned}
W_n(x+2)+F(W_n(x))+c(x)&=F(V_n(x+2))+c_s+c(x+2)+\sum_{h=0}^{k}q_{x,x-h}V_n(x+1-h)\\
&\quad+\sum_{h=k+1}^{x}q_{x,x-h}\left(F(V_n(x-h))+c_s+c(x-h)\right)+c(x).
\end{aligned}$$

Let us subtract this expression to the right hand side of Equation (6). Using $q_{x,x-h}=q_{x+1,x+1-h}$, for

$0 \leq h < x$, we get

$$
\begin{aligned}
&F(V_n(x+2)) + \cancel{c_s} + c(x+2) + \sum_{h=0}^{k} q_{x,x-h} V_n(x+1-h) + \sum_{h=k+1}^{x} q_{x,x-h} \left(F(V_n(x-h)) + c_s + c(x-h)\right) + c(x) \\
&\qquad - 2c(x+1) - F(V_n(x+1)) - \cancel{c_s} - \sum_{h=0}^{k} q_{x+1,x+1-h} V_n(x+2-h) \\
&\qquad - \sum_{h=k+1}^{x+1} q_{x+1,x+1-h} \left(F(V_n(x+1-h)) + c_s + c(x+1-h)\right) \\
&= c(x+2) + c(x) - 2c(x+1) + \cancel{F(V_n(x+1))} - \sum_{h=k+1}^{x+1} q_{x+1,x+1-h} V_n(x+1-h) - \cancel{F(V_n(x+1))} \\
&\qquad + \cancel{F(V_n(x+2))} + \sum_{h=k+1}^{x+2} q_{x+2,x+2-h} V_n(x+2-h) - \cancel{F(V_n(x+2))} + \sum_{h=k+1}^{x} q_{x,x-h} \left(F(V_n(x-h)) + c_s + c(x-h)\right) \\
&\qquad - \sum_{h=k+1}^{x+1} q_{x+1,x+1-h} \left(F(V_n(x+1-h)) + c_s + c(x+1-h)\right) \\
&= c(x+2) + c(x) - 2c(x+1) \\
&\quad + \sum_{h=k+1}^{x-1} q_{x+1,x+1-h} \left(V_n(x+2-h) + F(V_n(x-h)) + \cancel{c_s} + c(x-h) - V_n(x+1-h)\right. \\
&\qquad\qquad \left. - \left(F(V_n(x+1-h)) + \cancel{c_s} + c(x+1-h)\right)\right) \\
&\quad - (1-u)^{x+1} V_n(0) - u(1-u)^x V_n(1) + (1-u)^{x+2} V_n(0) + u(1-u)^{x+1} V_n(1) + u(1-u)^x V_n(2) \\
&\quad + (1-u)^x (V_n(0) + c_s + c(0)) - (1-u)^{x+1} (V_n(0) + c_s + c(0)) - u(1-u)^x (uV_n(1) + (1-u)V_n(0) + c_s + c(1)).
\end{aligned}
$$

The first term of the expression is positive since $c(x)$ is convex in x. The second term is also positive since V_n is *gcv*. The last two lines of the expression can be simplified into

$$u(1-u)^x \left(V_n(2) - 2uV_n(1) + (2u-1)V_n(0) + c(0) - c(1)\right).$$

We can further decompose this expression into

$$
\begin{aligned}
&V_n(2) - 2uV_n(1) + (2u-1)V_n(0) + c(0) - c(1) \\
&\quad = V_n(2) - (1+u)V_n(1) + uV_n(0) + c(0) - c(1) \\
&\qquad + (1-u)(V_n(1) - V_n(0)).
\end{aligned}
$$

The first line is positive since V_n is *gcv* and the second one is also positive since V_n is increasing in x. This proves that W_n is *gcv*.

From cases 1 and 2, we also deduce that

$$W_n(x+2) + F(W_n(x)) + c(x) - W_n(x+1) - F(W_n(x+1)) - c(x+1) \geq u(1-u)^{x+1}(V_n(1) - V_n(0)). \quad (7)$$

This inequality will be used to prove the next induction step.

V_n**,** W_n **increasing in** $x \Longrightarrow V_{n+1}$ **increasing in** x**.** We now assume that W_n and V_n are *igcv* and we prove that V_{n+1} is also *igcv*. We first prove that V_{n+1} is increasing in x. For $x = 0$, we have

$$\begin{aligned}
V_{n+1}(1) - V_{n+1}(0) &= \gamma W_n(1) - \lambda W_n(0) + \mu\left(F(V_n(1)) + c(1) - V_n(0)\right) + (1-\lambda-\mu)(V_n(1) - V_n(0)) \\
&\quad + (\lambda - \gamma)V_n(1) \\
&= \gamma(W_n(1) - W_n(0)) + \gamma(W_n(0) - V_n(0)) + \lambda(V_n(1) - W_n(0)) \\
&\quad + \mu u\left(V_n(1) - V_n(0)\right) + \mu c(1).
\end{aligned}$$

The first term proportional with γ is positive since W_n is increasing in x, the second term in γ is positive because either $W_n(0) = V_n(1)$ and $W_n(0) - V_n(0) = V_n(1) - V_n(0) \geq 0$ or $W_n(0) = V_n(0) + c_s + c(0)$ and $W_n(0) - V_n(0) = c_s + c(0) \geq 0$, the term in λ is also positive because $W_n(0) = \min(V_n(1), V_n(0) + c_s + c(0)) \leq V_n(1)$, the other terms are also clearly positive. Therefore, $V_{n+1}(1) \geq V_{n+1}(0)$. For $x > 0$, we have

$$\begin{aligned}
V_{n+1}(x+1) - V_{n+1}(x) &= \gamma(W_n(x+1) - W_n(x)) + \mu\left(F(V_n(x+1)) - F(V_n(x)) + c(x+1) - c(x)\right) \\
&\quad + (1-\gamma-\mu)(V_n(x+1) - V_n(x)) \geq 0.
\end{aligned}$$

Therefore V_{n+1} is increasing in x for $x \geq 0$.

V_n**,** W_n *gcv* $\Longrightarrow V_{n+1}$ *gcv***.** We now prove that V_{n+1} is generally convex. For $x = 0$, we may write

$$\begin{aligned}
&V_{n+1}(2) + uV_{n+1}(0) - (1+u)V_{n+1}(1) + c(0) - c(1) \\
&\quad = \gamma(W_n(2) - (1+u)W_n(1) + uW_n(0) + c(0) - c(1)) \\
&\qquad + (1-\gamma-\mu)(V_n(2) - (1+u)V_n(1) + uV_n(0) + c(0) - c(1)) \\
&\qquad + u\mu(V_n(2) - (1+u)V_n(1) + uV_n(0) + c(0) - c(1)) \\
&\qquad + u(1-u)\mu(V_n(1) - V_n(0)) \\
&\qquad + \mu(c(2) + c(0) - 2c(1)) \\
&\qquad - u\mu c(0) \\
&\qquad - (\gamma - \lambda)u(W_n(0) - V_n(0)).
\end{aligned}$$

The first three lines after the equality are positive since both W_n and V_n are *gcv*. The fourth line is also positive since V_n is increasing in x. The fifth line is positive since c is convex. The sixth line is equal to zero since $c(0) = 0$. The last line is negative. This however can be compensated by the first and the fourth lines.

Using Equation (7) and $W_n(0) \leq V_n(1)$, one may write

$$\begin{aligned}
&\gamma(W_n(2) - (1+u)W_n(1) + uW_n(0) + c(0) - c(1)) \\
&+ u(1-u)\mu(V_n(1) - V_n(0)) - (\gamma - \lambda)u(W_n(0) - V_n(0)) \\
&\quad \geq \gamma u(1-u)(V_n(1) - V_n(0)) + \mu u(1-u)(V_n(1) - V_n(0)) - u(\gamma - \lambda)(V_n(1) - V_n(0)) \\
&\quad = u(V_n(1) - V_n(0))\frac{\mu\gamma + \lambda^2}{\lambda + \gamma} \geq 0.
\end{aligned}$$

This proves that V_{n+1} is *gcv*.

For $x > 0$, we have

$$\begin{aligned}
F(V_{n+1}(x)) &= \sum_{h=0}^{x} q_{x,x-h} V_{n+1}(x-h) \\
&= \gamma F(W_n(x)) + (1-\gamma-\mu)F(V_n(x)) + \mu \sum_{h=0}^{x} q_{x,x-h}(F(V_n(x-h)) + c(x-h)) \\
&\quad + (1-u)^x(\gamma - \lambda)(V_n(0) - W_n(0)).
\end{aligned}$$

So,

$$\begin{aligned}
&V_{n+1}(x+2) + F(V_{n+1}(x)) + c(x) - V_{n+1}(x+1) - F(V_{n+1}(x+1)) - c(x+1) \\
&= \gamma(W_n(x+2) + F(W_n(x)) - W_n(x+1) - F(W_n(x+1)) + c(x) - c(x+1)) \\
&\quad + (1-\gamma-\mu)(V_n(x+2) + F(V_n(x)) - V_n(x+1) - F(V_n(x+1)) + c(x) - c(x+1)) \\
&\quad + \mu \sum_{h=0}^{x-1} u\,(1-u)^h\,(V_n(x+2-h) + F(V_n(x-h)) - V_n(x+1-h) - F(V_n(x+1-h)) \\
&\qquad\qquad + c(x-h) - c(x+1-h)) \\
&\quad + \mu(c(x+2) + c(x) - 2c(x+1)) \\
&\quad + \mu u(1-u)^x(V_n(2) + uV_n(1) - (1+u)V_n(0) + c(0) - c(1)) \\
&\quad + \mu u(1-u)^{x+1}(V_n(1) - V_n(0)) \\
&\quad - u(1-u)^x(\gamma - \lambda)(W_n(0) - V_n(0)).
\end{aligned} \tag{8}$$

The first three terms of the expression are positive since V_n and W_n are *gcv*. The fourth term is positive since $c(x)$ is convex in x. The fifth term again is positive since V_n is *gcv*. The sixth term is positive since V_n is increasing in x. Only the last term is negative. In what follows we show that this can be compensated by the first term in γ and the sixth term. Using Equation (7) and $W_n(0) \leq V_n(1)$, one may write

$$\begin{aligned}
&\gamma(W_n(x+2) + F(W_n(x)) - W_n(x+1) - F(W_n(x+1)) + c(x) - c(x+1)) \\
&+ \mu u(1-u)^{x+1}(V_n(1) - V_n(0)) - u(1-u)^x(\gamma - \lambda)(W_n(0) - V_n(0)) \\
&\quad \geq \gamma u(1-u)^{x+1}(V_n(1) - V_n(0)) + \mu u(1-u)^{x+1}(V_n(1) - V_n(0)) - u(1-u)^x(\gamma - \lambda)(V_n(1) - V_n(0)) \\
&\qquad = u(1-u)^x(V_n(1) - V_n(0))\frac{\mu\gamma + \lambda^2}{\lambda + \gamma} \geq 0.
\end{aligned}$$

This proves that V_{n+1} is *gcv*. □

Proof of Theorem 2

Stationary probabilities. Observing that

$$\left(\frac{\gamma}{\lambda+\gamma}\right)^x + \sum_{l=h}^{x-1}\left(\frac{\lambda}{\lambda+\gamma}\right)\left(\frac{\gamma}{\lambda+\gamma}\right)^l = \left(\frac{\gamma}{\lambda+\gamma}\right)^h, \tag{9}$$

we deduce that the cumulative transition rate from state x to states $0, 1, \cdots x-h$ is $\mu\left(\frac{\gamma}{\lambda+\gamma}\right)^h$, for $0 \leq h < x < n$; and that from state n to states $0, 1, \cdots n-h$ is $(\mu+\gamma)\left(\frac{\gamma}{\lambda+\gamma}\right)^h$, for $0 \leq h < n$. We now give the steady-state probability to be in state x, denoted by p_x, for $0 \leq x \leq n$.

Lemma 1 *We have*

$$\begin{aligned}
p_0 &= \frac{\gamma(\mu - \lambda)}{\gamma\mu + \lambda^2 - \lambda(\lambda+\gamma)\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^n}, \\
p_x &= \frac{\lambda}{\gamma}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^x p_0, \ \textit{for } 0 < x \leq n.
\end{aligned}$$

Proof. We first prove by induction on x that

$$p_{n-x} = \left(\frac{\mu+\gamma}{\lambda+\gamma}\right)^x p_n, \tag{10}$$

for $0 \leq x < n$. For $x = 0$, Equation (10) is straightforward. Assume now that Equation (10) is true for any rank r such that $0 \leq l \leq x$ and $0 \leq x < n-1$. Using Equation (9), we may write

$$\begin{aligned}
\gamma p_{n-(x+1)} &= \mu\sum_{l=1}^{x}\left(\frac{\gamma}{\lambda+\gamma}\right)^{x+1-l} p_{r-l} + (\mu+\gamma)\left(\frac{\gamma}{\gamma+\lambda}\right)^{x+1} p_n \\
&= \mu\sum_{l=1}^{x}\left(\frac{\gamma}{\lambda+\gamma}\right)^{x+1-l}\left(\frac{\mu+\gamma}{\lambda+\gamma}\right)^l p_n + (\mu+\gamma)\left(\frac{\gamma}{\lambda+\gamma}\right)^{x+1} p_n \\
&= (\mu+\gamma)\left(\frac{\gamma}{\lambda+\gamma}\right)^{x+1}\left(\frac{\mu+\gamma}{\gamma}\right)^x p_n.
\end{aligned}$$

This leads to $p_{n-(x+1)} = \left(\frac{\mu+\gamma}{\lambda+\gamma}\right)^{x+1} p_n$ and finishes the proof of Equation (10). Note now that for p_0, the transition rate from state 0 to 1 is λ instead of γ. Thus

$$p_0 = \frac{\gamma}{\lambda}\left(\frac{\mu+\gamma}{\lambda+\gamma}\right)^n p_n. \tag{11}$$

Since all probabilities sum up to one, we obtain p_0. This finishes the proof of the lemma. □

Probability of an empty system. If $\lambda \neq \mu$, we have $\frac{\lambda+\gamma}{\mu+\gamma} = \frac{\lambda+n/\mathrm{ADT}}{\mu+n/\mathrm{ADT}} = \left(1+\frac{\lambda}{n/\mathrm{ADT}}\right)\frac{1}{1+\frac{\mu}{n/\mathrm{ADT}}}$. As n tends to ∞, $\frac{1}{1+\frac{\mu}{n/\mathrm{ADT}}} = 1 - \frac{\mu}{n/\mathrm{ADT}} + o(1/n)$. Thus as n tends to ∞, $\frac{\lambda+\gamma}{\mu+\gamma} = 1 + \frac{\lambda-\mu}{n/\mathrm{ADT}} + o(1/n)$. We also have as n tends to ∞, $\ln\left(\frac{\lambda+\gamma}{\mu+\gamma}\right) = \frac{\lambda-\mu}{n/\mathrm{ADT}} + o(1/n)$. Then, $\lim_{n\to\infty} n\ln\left(\frac{\lambda+\gamma}{\mu+\gamma}\right) = -\mathrm{ADT}(\mu-\lambda)$, which implies $\lim_{n\to\infty}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^n = e^{-\mathrm{ADT}(\mu-\lambda)}$. If $\lambda = \mu$, we have $\frac{\lambda+\gamma}{\mu+\gamma} = 1$. Then, $\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^n = 1$, for $r \geq 1$ and we also have $\lim_{n\to\infty}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^n = e^{-\mathrm{ADT}(\mu-\lambda)}$. We therefore deduce the result of the proposition by letting n and γ go to infinity.

Other performance measures. We prove the equations for the case $\lambda \neq \mu$. The proofs for the case $\lambda = \mu$ follow from those of the case $\lambda \neq \mu$ by continuity. We consider the embedded Markov chain at matching initiation or sent back to shipping line epochs. Matching initiations occur at μ-transitions from states $x > 0$. Sent back to the shipping line initiations only occur in state n with a γ-transition. The state probability just before a match or a send back to the shipping line is denoted by $\alpha(x)$. From flow conservation, we may write, $\alpha(x) = \frac{\mu p_x}{\lambda}$ for $0 < x < 0$, and $\alpha(n) = \frac{(\gamma+\mu)p_n}{\lambda}$ for $x = n$.
Proportion of containers sent back to the shipping line: It is given by $P_s = \lim_{n\to\infty}\left(\frac{\gamma}{\lambda}p_n\right)$.
Expected time spent by container at the consignee: A matched container waits x γ-phases with probability $p_x\frac{\mu}{\lambda}$, for $0 < x \leq n$. We denote by T_S the time spent by a container which is sent to the shipper at the consignee. Averaging over all possibilities, we obtain

$$\begin{aligned}
(1-P_s)\cdot E(T_S) &= \lim_{n\to\infty} \sum_{x=1}^{n} \frac{\mu}{\lambda}\frac{x}{\gamma}\frac{\lambda}{\gamma}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^x p_0 \\
&= \lim_{n\to\infty} \frac{\mu}{\gamma^2}\sum_{x=1}^{n} x\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^x p_0 \\
&= \lim_{n\to\infty} \frac{\mu}{\gamma^2}\frac{\lambda+\gamma}{\mu+\gamma}\frac{-(n+1)(1-\frac{\lambda+\gamma}{\mu+\gamma})\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^n + 1 - \left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{n+1}}{\left(1-\frac{\lambda+\gamma}{\mu+\gamma}\right)^2} p_0.
\end{aligned}$$

We therefore obtain $(1-P_s)\cdot E(T_S) = \frac{1-e^{-\mathrm{ADT}(\mu-\lambda)}(1+\mathrm{ADT}(\mu-\lambda))}{\mu(1-a)(1-ae^{-\mathrm{ADT}(\mu-\lambda)})}$. Finally, $E(T) = (1-P_s)\cdot E(T_S) + P_s\cdot\mathrm{ADT}$.
Waiting time distribution: A matched container can wait x γ-phases with probability $\frac{\mu}{\lambda}p_x$, for $0 < x \leq n$. For a container that matches from state $x > 0$, its waiting time is an Erlang random variable with x phases

and a rate γ per phase. We thus have

$$
\begin{aligned}
(1-P_s)\cdot P(T_S>t) &= \lim_{n\to\infty} \sum_{x=1}^{n}\frac{\mu}{\lambda}\frac{\lambda}{\gamma}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{x} p_0 \sum_{h=0}^{x-1}\frac{(\gamma t)^h}{h!}e^{-\gamma t}\\
&= \lim_{n\to\infty} \frac{\mu}{\lambda}\frac{\lambda}{\gamma}p_0 e^{-\gamma t}\sum_{x=1}^{n}\sum_{h=0}^{x-1}\frac{(\gamma t)^h}{h!}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{x}.
\end{aligned}
$$

We also may write

$$
\begin{aligned}
\sum_{x=1}^{n}\sum_{h=0}^{x-1}\frac{(\gamma t)^h}{h!}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{x} &= \frac{\lambda+\gamma}{\mu+\gamma}\sum_{h=0}^{n-1}\frac{(\gamma t)^h}{h!}\sum_{x=h}^{n-1}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{x}\\
&= \frac{\lambda+\gamma}{\mu+\gamma}\sum_{h=0}^{n-1}\frac{(\gamma t)^h}{h!}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{h}\frac{1-\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{n-1-(h-1)}}{1-\frac{\lambda+\gamma}{\mu+\gamma}}\\
&= \frac{\lambda+\gamma}{\mu-\lambda}\left(\sum_{h=0}^{n-1}\frac{(\gamma t)^h}{h!}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{h}-\sum_{h=0}^{n-1}\frac{(\gamma t)^h}{h!}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{n}\right).
\end{aligned}
$$

Next, we get

$$
\lim_{n\to\infty} e^{-\gamma t}\sum_{h=0}^{n-1}\frac{(\gamma t)^h}{h!}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{n} = e^{-\mathrm{ADT}(\mu-\lambda)},\ \text{and}\ \lim_{n\to\infty} e^{-\gamma t}\sum_{h=0}^{n-1}\frac{(\gamma t)^h}{h!}\left(\frac{\lambda+\gamma}{\mu+\gamma}\right)^{h} = e^{-t(\mu-\lambda)}.
$$

Therefore $(1-P_s)\cdot P(T_S>t)=\frac{e^{-t(\mu-\lambda)}-e^{-\mathrm{ADT}(\mu-\lambda)}}{1-ae^{-\mathrm{ADT}(\mu-\lambda)}}$ Using $P(T>t)=(1-P_s)\cdot P(T_S>t)+P_s$, we next obtain the result. $\square$

Proof of Proposition 1 and Corollary 2

Proof of Proposition 1. We compute the first and second order derivatives of the performance measures so as to evaluate their sign. Since all these measures are infinitely continuously derivable in ADT for ADT >0, the strict convexity of the performance measures in the case $\lambda\neq\mu$ implies the convexity in the case $\lambda=\mu$. Moreover, note that the strict convexity for $\lambda=\mu$ holds also by following exactly the same approach as that for $\lambda\neq\mu$. In what follows, we focus on the case $\lambda\neq\mu$.

Let us first prove that $p_0(\infty)$ is decreasing and convex in ADT. In what follows we write p_0 instead of $p_0(\infty)$. We may write

$$
\frac{\partial p_0}{\partial \mathrm{ADT}} = -\frac{\lambda(1-a)^2e^{-\mathrm{ADT}(\mu-\lambda)}}{\left(1-ae^{-\mathrm{ADT}(\mu-\lambda)}\right)^2}<0.
$$

Thus p_0 is strictly decreasing in ADT. One may write

$$
\frac{\partial^2 p_0}{\partial \mathrm{ADT}^2} = \frac{\lambda\mu(1-a)^3e^{-\mathrm{ADT}(\mu-\lambda)}\left(1+ae^{-\mathrm{ADT}(\mu-\lambda)}\right)}{\left(1-ae^{-\mathrm{ADT}(\mu-\lambda)}\right)^3}.
$$

The sign of $\frac{\partial^2 p_0}{\partial \mathrm{ADT}^2}$ only depends on $\frac{1-a}{1-ae^{-\mathrm{ADT}(\mu-\lambda)}}$ because the other parts of the expression are positive. If

$1-a>0$, then $\text{ADT}(\mu-\lambda)>0$. So $e^{-\text{ADT}(\mu-\lambda)}<1$ and $1-ae^{-\text{ADT}(\mu-\lambda)}>0$. Similarly, if $1-a<0$, then $1-ae^{-\text{ADT}(\mu-\lambda)}<0$. In both cases the sign of the ratio is positive. Therefore, $\frac{\partial^2 p_0}{\partial \text{ADT}^2}>0$ and p_0 is strictly decreasing and strictly convex in ADT.

We next focus on the proportion of containers sent back to the shipping line, P_s. We have

$$\begin{aligned}\frac{\partial P_s}{\partial \text{ADT}} &= e^{-\text{ADT}(\mu-\lambda)}\left(-(\mu-\lambda)p_0+\frac{\partial p_0}{\partial \text{ADT}}\right)\\ &= -\frac{\mu(1-a)^2 e^{-\text{ADT}(\mu-\lambda)}}{\left(1-ae^{-\text{ADT}(\mu-\lambda)}\right)^2}<0.\end{aligned}$$

Therefore $\frac{\partial P_s}{\partial \text{ADT}}<0$, and P_s is strictly decreasing in ADT. Observe that

$$\frac{\partial P_s}{\partial \text{ADT}} = a^{-1}\frac{\partial p_0}{\partial \text{ADT}}.$$

Thus, $\frac{\partial P_s}{\partial \text{ADT}}$ is positively proportional to $\frac{\partial p_0}{\partial \text{ADT}}$. Since p_0 is strictly convex in ADT, P_s is also strictly convex in ADT.

We next focus on $E(T)$ and $E(N)$. We have $E(T)=\frac{1-e^{-X}(1+aX)}{\mu(1-a)(1-ae^{-X})}$, with $X=\text{ADT}(\mu-\lambda)$. This allows us to derive $E(T)$. We have

$$\frac{\partial E(T)}{\partial \text{ADT}} = \frac{\partial E(T)}{\partial X}\cdot\frac{\partial X}{\partial \text{ADT}}.$$

After some algebra, we get

$$\frac{\partial E(T)}{\partial \text{ADT}} = \frac{e^{-X}(1-2a+aX+a^2e^{-X})}{(1-ae^{-X})^2}.$$

The sign of this expression depends on the sign of $1-2a+aX+a^2e^{-X}$. We have $e^{-X}=\sum_{k=2}^{\infty}\frac{(-\mu\text{ADT}(1-a))^k}{k!}+1-\mu\text{ADT}(1-a)$. So, $1-2a+aX+a^2e^{-X}=(1-a)^2(1+a\mu\text{ADT})+a^2\sum_{k=2}^{\infty}\frac{(-X)^k}{k!}$. This expression is clearly positive because $\sum_{k=2}^{\infty}\frac{(-X)^k}{k!}=e^{-X}-1+X\geq 0$ for $X\in\mathbb{R}$.

It remains to prove that $P(T>t)$ is strictly increasing in ADT for $t<\text{ADT}$. One may write

$$\frac{\partial P(T>\text{ADT})}{\partial \text{ADT}} = \frac{\lambda e^{-\mu\text{ADT}(1-a)}(1-a)(1-e^{-t\mu(1-a)})}{(1-ae^{-\mu\text{ADT}(1-a)})^2}.$$

This sign of this expression depends on the sign of $(1-a)(1-e^{-t\mu(1-a)})$. With the same approach as for p_0 by distinguishing the cases $a>1$ and $a<1$, one can show hat this expression is positive. So, $P(T>\text{ADT})$ is strictly increasing in ADT for $t<\text{ADT}$.

Let us now consider the expected cost per container, $E(C)$. We assume that $c(t)$ is a continuous piecewise linear increasing in t. We denote by c_k and d_k, the slope and the intercept of the function $c(t)$ on the interval $[t_{k-1},t_k)$ with the convention that $t_0=0$ for $k\geq 1$. The cost function is convex, therefore c_k is increasing in k. The parameters d_k are adjusted in order to make $c(t)$ a continuous function. As a function of the time spent at the consignee, the cost function, $c(t)$, can be written as: $c(t) = c_k \cdot t + d_k$ for $t \in [t_{k-1}, t_k)$ and $k \geq 1$. For ADT $\in [t_{k-1}, t_k)$, using the result of Corollary 1, $E(C)$ is given by

$$E(C) = (c_s + c_k\mathrm{ADT} + d_k)\frac{(1-a)e^{-\mathrm{ADT}(\mu-\lambda)}}{1-ae^{-\mathrm{ADT}(\mu-\lambda)}} + \int_{t=0}^{\mathrm{ADT}} \mu\frac{(1-a)e^{-t(\mu-\lambda)}}{1-ae^{-\mathrm{ADT}(\mu-\lambda)}} \cdot (c_k t + d_k)\,\mathrm{d}t.$$

After some algebra, this leads to

$$E(C) = \frac{c_k + d_k\mu(1-a) + e^{-\mu\mathrm{ADT}(1-a)}\left((1-a)^2(c_s + c_k\mathrm{ADT} + d_k)\mu - (1-a)\mu(c_k\mathrm{ADT} + d_k) - c_k\right)}{\mu(1-a)(1-ae^{-\mu\mathrm{ADT}(1-a)})},$$

for ADT $\in [t_{k-1}, t_k)$. We now compute the derivative of this function in ADT:

$$\frac{\partial E(C)}{\partial \mathrm{ADT}} = \frac{e^{-\mu\mathrm{ADT}(1-a)}\left((1-a)^2c_s\mu - c_k(1-2a+\mu\mathrm{ADT}a(1-a) + a^2e^{-\mu\mathrm{ADT}(1-a)})\right)}{(1-ae^{-\mu\mathrm{ADT}(1-a)})^2},$$

for ADT $\in [t_{k-1}, t_k)$. Note that the parameter d_k is not present is this expression therefore it has no influence on the optimal threshold. The sign of $\frac{\partial E(C)}{\partial \mathrm{ADT}}$ depends on the sign of the function in ADT, $f(\mathrm{ADT}) = (1-a)^2c_s\mu - c_k(1-2a+\mu\mathrm{ADT}a(1-a) + a^2e^{-\mu\mathrm{ADT}(1-a)})$. This function is defined for ADT ≥ 0. By deriving again the function f, we can prove that this function is strictly decreasing in ADT. Thus if $c_s\mu \leq c_k$, $\frac{\partial E(C)}{\partial \mathrm{ADT}} \geq 0$ for ADT $\in [t_{k-1}, t_k)$ and the optimal threshold on the interval $[t_{k-1}, t_k)$ is ADT $= t_{k-1}$. Otherwise, $\frac{\partial E(C)}{\partial \mathrm{ADT}}$ is first negative, next positive. Let us denote by ADT_k the unique solution of $f(\mathrm{ADT}) = 0$ for ADT ≥ 0. If $\mathrm{ADT}_k < t_{k-1}$, then the optimal threshold on the interval $[t_{k-1}, t_k)$ is ADT $= t_{k-1}$. If $t_{k-1} \leq \mathrm{ADT}_k < t_{k-1}$, then the optimal threshold on the interval $[t_{k-1}, t_k)$ is ADT $= \mathrm{ADT}_k$. Finally, if $\mathrm{ADT}_k \geq t_k$, then the optimal threshold on the interval $[t_{k-1}, t_k)$ is ADT $= t_k$. This allows us to find the optimal threshold on the interval $[t_{k-1}, t_k)$.

We are now interested in obtaining the global optimal threshold on the interval $[0, \infty)$. We now show that ADT_k is decreasing in k. We use the equality which allows to obtain ADT in order to find $\frac{\partial \mathrm{ADT}}{\partial c_k}$. One may write

$$\frac{\partial \mathrm{ADT}}{\partial c_k} = -\frac{c_s(1-a)}{c_k^2 a(1-ae^{-\mu(1-a)\mathrm{ADT}})} < 0.$$

Therefore, ADT is decreasing and convex in c_k. Since c_k is increasing in k then ADT_k is decreasing in k. Hence, the optimal threshold is either the first value ADT_k, for which $\mathrm{ADT}_k \in [t_{k-1}, t_k)$ and $f(\mathrm{ADT}_k) = 0$. Otherwise the two conditions $\mathrm{ADT}_k \in [t_{k-1}, t_k)$ and $f(\mathrm{ADT}_k) = 0$ cannot be met together, therefore there exists an index k such that $E(C)$ is decreasing for $t \leq t_{k-1}$ and increasing otherwise. The optimal threshold is hence t_{k-1}.

Proof of Corollary 2. Using proposition 1, we use Relation 2 to find $\frac{\partial \mathrm{ADT}}{\partial c_s}$ and $\frac{\partial \mathrm{ADT}}{\partial c_k}$. One may write

$$\begin{aligned}
\frac{\partial \mathrm{ADT}}{\partial c_s} &= \frac{1-a}{c_k a(1-ae^{-\mu(1-a)\mathrm{ADT}})} > 0,\\
\frac{\partial^2 \mathrm{ADT}}{\partial c_s^2} &= -\frac{\partial \mathrm{ADT}}{\partial c_s}\frac{a\mu(1-a)^2 e^{-\mu(1-a)\mathrm{ADT}}}{c_k a(1-ae^{-\mu(1-a)\mathrm{ADT}})^2} < 0,\\
\frac{\partial^2 \mathrm{ADT}}{\partial c_k^2} &= c_s\frac{2c_k a(1-a)(1-ae^{-\mu(1-a)\mathrm{ADT}}) + c_k^2 a^2\mu(1-a)^2 e^{-\mu(1-a)\mathrm{ADT}}}{(c_k^2 a(1-ae^{-\mu(1-a)\mathrm{ADT}}))^2} > 0.
\end{aligned}$$

Therefore, ADT is increasing and piecewise concave in c_s and decreasing and piecewise convex in c_k. The other results in Corollary 2 follow directly from combining Proposition 1 and the monotonicity results of ADT in c_s and c_k. □